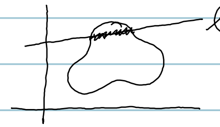
Förelåsning 8/10-13

Fouriertransformer

intro: Problem:



Låt $K \subseteq \mathbb{C}$

Antag att $\forall\ \ell \subseteq \mathbb{C}$ linjer

så vet vi $|\ell \cap K|$. Kan vi rekonstruera K?

Ja, Radontransformen (använder Fouriertransf.)

Tomograf: $|\ell \cap K|$ = absorbtionen av en (röntgen)stråle genom K, längs med ℓ.

Definition

$u \in L^1(\mathbb{R})$, $\displaystyle\int_{-\infty}^{\infty} |u(t)|\,dt < \infty$

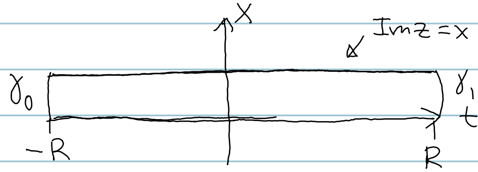
$$\hat{u}(x) = \int_{-\infty}^{\infty} \underbrace{e^{-itx} u(t)}_{|-\shortparallel-| \le |u|}\,dt = \mathcal{F}(u)(x)$$

ex $u(t) = e^{-t^2/2}$

Enligt def. har vi

$$\hat{u}(x) = \int_{-\infty}^{\infty} e^{-t^2/2 - itx}\,dt = \underbrace{\int_{-\infty}^{\infty} e^{-(\overbrace{t+ix}^{z})^2/2}\,dt}_{I}\; e^{-x^2/2}$$

$$I = \int_{\operatorname{Im} z = x} e^{-z^2/2}\,dz \quad \text{ty} \begin{cases} z = t + ix \\ dz = dt \end{cases}$$



$$\int_{\Gamma_R} e^{-z^2/2}\,dz = 0$$

forts. →

Cauchy $\Longrightarrow \displaystyle\int_{-R}^{R} e^{-t^2/2}\,dt - \int_{\substack{\mathrm{Im}\, z \\ -R<t<R}} = \int_{\gamma_1} - \int_{\gamma_0} \xrightarrow[R\to\infty]{} 0$

$\therefore \displaystyle\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = I = \sqrt{2\pi}$

$\therefore \hat{u}(x) = \sqrt{2\pi}\, e^{-x^2/2}$, $\quad (u = e^{-t^2/2})$

(annat bevis senare)

<u>Kom ihåg:</u> Inversionsformel:

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx}\, \hat{u}(x)\, dx$$

I vårt fall: $e^{-t^2/2} = \dfrac{\sqrt{2\pi}}{2\pi} \displaystyle\int e^{itx} e^{-x^2/2}\, dx =$

$$= \frac{\sqrt{2\pi}\sqrt{2\pi}}{2\pi} e^{-t^2/2}$$

<u>Egenskaper hos $\mathcal{F}$-transformer.</u>

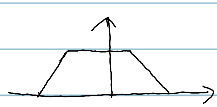
a) $\mathcal{F}(au+bv) = a\hat{u} + b\hat{v}$ om $a, b \in \mathbb{C}$

b) $a \in \mathbb{R}$, $\widehat{u(t+a)} = e^{iax}\, \hat{u}(x)$

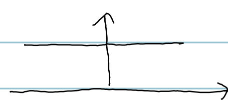
c) $b \in \mathbb{R}$, $b \neq 0$, $\widehat{u(bt)} = \dfrac{1}{|b|}\, \hat{u}\left(\dfrac{x}{b}\right)$

d) $\mathcal{F}(u')(x) = (ix)\, \hat{u}(x)$, gäller <u>om</u>
u kont. och u' styckvis kont.

<u>ex</u>



OK!



<u>ej</u> OK!

e) om $\displaystyle\int_{-\infty}^{\infty} |t||u|\,dt < \infty$ så $\dfrac{d\hat{u}}{dx} = \widehat{-itu}$, $\; i\dfrac{d\hat{u}}{dx} = \widehat{tu}$

ex $\quad u(t) = \dfrac{1}{t^2 + 2t + 5}$

(Kom ihåg: $\widehat{\dfrac{1}{1+t^2}} = \pi e^{-|x|}$)

$u(t) = \dfrac{1}{(t+1)^2 + 4}$

Låt $u_1(t) = \dfrac{1}{t^2+4}$

enligt $(b=2)$: $\hat{u}(x) = e^{ix}\,\hat{u}_1$

Men $u_1(t) = \dfrac{1}{4}\,\dfrac{1}{(t/2)^2+1}$. Enligt $(3=c)$,

$\hat{u}_1(x) = \dfrac{1}{4}\pi e^{-|2x|}\cdot 2 = \dfrac{\pi}{2} e^{-2|x|}$

$\therefore\ \hat{u}(x) = \dfrac{\pi}{2} e^{ix-2|x|}$

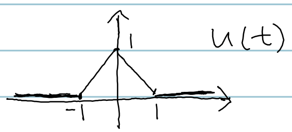
ex $\quad u(t) = e^{-t^2}$

Vet att $\mathcal{F}(e^{-t^2/2}) = \sqrt{2\pi}\, e^{-x^2/2}$

$u(t) = e^{-(\sqrt{2}t)^2/2}$

$\therefore\ \hat{u}(x) = \dfrac{\sqrt{2\pi}}{\sqrt{2}} e^{-(x/\sqrt{2})^2/2} = \sqrt{\pi}\, e^{-x^2/4}$

ex



Kont. & styckvis deriv.bar.

$\mathcal{F}(u') = ix\mathcal{F}(u)$, $\therefore\ \mathcal{F}(u) = \dfrac{1}{ix}\mathcal{F}(u')$

$\mathcal{F}(u') = \displaystyle\int_{-1}^{0} e^{-itx}\,dt - \int_{0}^{1} e^{-tx}\,dt = \dfrac{2\cos x - 2}{ix}$

forts. →

$\therefore\ \mathcal{F}(u) = \frac{1}{(ix)^2}(2\cos x - 2) = 2 \cdot \frac{1-\cos x}{x^2}$

$$\hat{u}(0) = 2 \lim_{x\to 0} \frac{1-(1-x^2/2+\dots)}{x^2} = 1.$$

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(t)\,dt = 1$$

Stämmer! Visar att man har räknat rätt.

Bevis (av [b(=2)])

$$\mathcal{F}(u(t+a))(x) = \int_{-\infty}^{\infty} u(t+a)e^{-itx}\,dt = \{t+a=s\} =$$

$$= \int_{-\infty}^{\infty} u(s)\,e^{-i(s-a)x}\,ds = e^{iax}\int_{-\infty}^{\infty} u(s)e^{-isx}\,ds = e^{iax}\,\hat{u}(x).$$

d) $\mathcal{F}(u')(x) = \left\{\begin{array}{l}\text{antag } u' \text{ kont.}\\ u(t)\to 0 \text{ då } t\to\pm\infty\end{array}\right\} =$

$$= \lim_{R\to\infty}\int_{-R}^{R} u'(t)\,e^{-itx}\,dt = \lim_{R\to\infty}\left(\left[u(t)e^{-itx}\right]_{-R}^{R} - \int_{-R}^{R} u(t)(ix)e^{-itx}\,dt\right)$$

$$= 0 - 0 + ix\int_{-\infty}^{\infty} u(t)\,e^{-itx}\,dt = ix\,\hat{u}(x).$$

Formellt: $\mathcal{F}\left(\frac{u(t+a)-u(t)}{a}\right) = \frac{e^{iax}-1}{a}\,\hat{u}$

$a\to 0$ ger formellt $\mathcal{F}(u') = ix\,\hat{u}$

Faltning

Låt $u, v \in L^1$ $\left(\int|u|, \int|v| < \infty\right)$

Definition

$$u * v(t) = \int_{-\infty}^{\infty} u(t-s)v(s)\,ds$$

$$\int_{-\infty}^{\infty} |u*v(t)|\,dt \leq \iint |u(t-s)||v(s)|\,dt\,ds$$

Men $\int |u(t-s)|\,dt = \int |u(t)|\,dt$

$\therefore \quad \int |u*v|\,dt \leq \int |u|\,dt \int |v|\,ds < \infty$

$\therefore \quad u*v \in L^1$

Proposition: $\widehat{u*v} = \hat{u}\cdot\hat{v}$

Följd: $u*v = v*u$

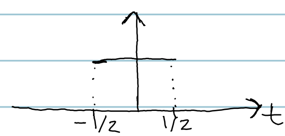
Bevis

$$\widehat{u*v}(x) = \int u*v(t)\,e^{-itx}\,dt = \iint u(t-s)v(s)e^{-itx}\,dt\,ds =$$

$$= \int v(s)\Big(\underbrace{\int u(t-s)e^{-itx}\,dt}_{t-s=y}\Big)ds = \int v(s)\int u(y)e^{-ix(y+s)}\,dy\,ds$$

$$= \int v(s)e^{-ixs}\,ds \int u(y)e^{-ixy}\,dy = \hat{v}(x)\,\hat{u}(x) \quad \blacksquare$$

ex $\quad v(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$



$$v*v(t) = \int_{-\infty}^{\infty} v(t-s)v(s)\,ds = \int_{-1/2}^{1/2} v(t-s)\,ds$$

forts. →

(i) $t>1 \Rightarrow v*v = 0$

(ii) $0<t<1 \Rightarrow v*v = 1-t$

symmetri $\Rightarrow$

$u(t) = v*v$

$$\hat{u}(x) = \frac{2-2\cos x}{x^2}$$

($\hat{u}(x)$ räknades ut tidigare idag)

Å andra sidan,

$$\hat{v}(x) = \int_{-1/2}^{1/2} e^{-itx} = \left[\frac{e^{-itx}}{ix}\right]_{-1/2}^{1/2} = \frac{e^{ix/2} - e^{-ix/2}}{ix} = \frac{2\sin(x/2)}{x}$$

$$\therefore \widehat{v*v} = \frac{4\sin^2(x/2)}{x^2} = \hat{u}(x)$$

ex Låt $u(t,s)$ lösa $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2}$, $t>0$, $s\in\mathbb{R}$

$u(t,s)$ = värmen vid punkt s vid $t>0$.

$u(0,s) = f(s)$ (Begynnelsevillkor).

Fouriertransf. i s.

$$\phi(t,x) = \int_{-\infty}^{\infty} u(t,s)\, e^{-isx}\, ds$$

$$-x^2\phi(t,x) = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial s^2} e^{-isx}\, ds = \int \frac{\partial u}{\partial t} e^{-isx}\, ds =$$

$$= \frac{\partial}{\partial t}\int u e^{-isx}\, ds = \frac{\partial \phi}{\partial t}$$

$$\therefore \frac{\partial \phi}{\partial t} = -x^2\phi$$ bara <u>en</u> derivata

$$\phi(0,x) = \hat{f}(x)$$

$$\Rightarrow \phi(t,x) = e^{-tx^2}\cdot C(x) = \hat{f}(x)\, e^{-tx^2}.$$