Storgruppsövning 16/10-13

5.1 Fouriertransformer

Definition

Antag $u: \mathbb{R} \to \mathbb{C}$ s.a $\int_{-\infty}^{\infty} |u(t)|\,dt < \infty \overset{\text{def.}}{\iff} u \in L^1$

Bilda en ny fkn $\hat{u}(x)$, $\mathcal{F}[u(t)](x) : \mathbb{R} \to \mathbb{C}$ genom

$$\hat{u}(x) = \int_{-\infty}^{\infty} u(t) e^{-itx}\,dt \left(\leq \int_{-\infty}^{\infty} |u(t)|\,dt < \infty \right)$$

$\mathcal{F}(u)$, $\hat{u}$ kallas Fouriertransformen av u.

ex (i) $u(t) = \chi_a(t) = \begin{cases} 1 & \text{om } |t| < a \\ 0 & \text{om } |t| > a \end{cases}$

$$\Rightarrow \mathcal{F}(\chi_a)(x) = \frac{2\sin(ax)}{x}$$

(ii) $u(t) = e^{-t^2} \Rightarrow \hat{u}(x) = \sqrt{\pi}\, e^{-x^2/4}$

(iii) $u(t) = \frac{1}{1+t^2} \Rightarrow \hat{u}(x) = \pi e^{-|x|}$

Egenskaper

(I) $\mathcal{F}$ linjär, dvs $u_1, u_2 \in L^1$, $\lambda_1, \lambda_2 \in \mathbb{C}$

$$\Rightarrow \mathcal{F}(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 \mathcal{F}(u_1) + \lambda_2 \mathcal{F}(u_2)$$

(II) $u \in L^1$, $a \in \mathbb{R}$

$$\Rightarrow \mathcal{F}(u(t+a))(x) = e^{iax}\hat{u}(x)$$

(III) $u \in L^1,\ b \in \mathbb{R} \setminus \{0\}$

$$\Rightarrow \mathcal{F}(u(bt))(x) = 1/|b| \cdot \hat{u}(x/b)$$

(IV) $u \in L^1$, u kont., $\lim_{h \to 0^{\pm}} u'(x+h)$ existerar $\forall x \in \mathbb{R}$, $u' \in L^1$

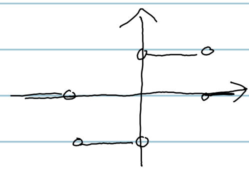
$$\Rightarrow \mathcal{F}(u'(t))(x) = ix\hat{u}(x)$$

(V) $u, v \in L^1$, Låt $(u * v)(t) = \int_{-\infty}^{\infty} u(s)v(t-s)\,ds$

$$\Rightarrow \mathcal{F}(u * v)(x) = \hat{u}(x)\hat{v}(x)$$

5.1.2

Finn $\hat{u}(x)$ om $u(t) = \begin{cases} -1 & -6 < t < 0 \\ 1 & 0 < t < 6 \\ 0 & |t| > 6 \end{cases}$



lösn.:

$$\hat{u}(x) = \int_{-\infty}^{\infty} u(t) e^{-ixt}\,dt = \int_{-6}^{0} -1 \cdot e^{-ixt}\,dt + \int_{0}^{6} 1 \cdot e^{-ixt}\,dt =$$

$$= -\left[\frac{e^{-ixt}}{-ix}\right]_{t=-6}^{t=0} + \left[\frac{e^{-ixt}}{-ix}\right]_{t=0}^{t=6} = \ldots =$$

$$= \frac{2(1-\cos(6x))}{ix} \quad \text{då } x \neq 0$$

$x=0$: $\hat{u}(0) = \int_{-\infty}^{\infty} u(t)\,dt = \int_{6}^{0} -1\,dt + \int_{0}^{6} 1\,dt = 0$

5.2 Egenskaper för $\mathcal{F}$

1) Sats: (Fouriers inversionsformel) — FIF

Antag $u \in L^1$ s.a $\hat{u}$ väldef.

Om $\hat{u} \in L^1$ så gäller $u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(x) e^{ixt} dt$

Problem: $u \in L^1$ medför ej nödvändigtvis $\hat{u} \in L^1$.

ex $u = \chi_a \in L^1 \implies \hat{u}(x) = \frac{2\sin(ax)}{x} \notin L^1$

Definition: $L^2 = \{u : \mathbb{R} \to \mathbb{C} \,;\, \int_{-\infty}^{\infty} |u(t)|^2 dt < \infty\}$

2) Sats: Fouriertr., egenskaper I–IV, samt FIF "funkar" med L^1 utbytt mot L^2

3) Sats: $u, v \in L^2$

(i) $2\pi \int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{u}(x)|^2 dx$. Parseval

(ii) $2\pi \int_{-\infty}^{\infty} u(t)\overline{v(t)} = \int_{-\infty}^{\infty} \hat{u}(x)\overline{\hat{v}(x)} dx$ Plancherel

Obs (i) ger att $u \in L^2 \iff \hat{u} \in L^2$

5.2.1

Visa att $\int_{-\infty}^{\infty} \frac{\sin(\alpha x)\sin(\beta x)}{x^2} = \pi \min(\alpha, \beta)$, $\alpha, \beta > 0$

Bevis

Låt $u(t) = \chi_\alpha(t)$, $v(t) = \chi_\beta$. Då gäller

$\hat{u}(x) = \frac{2\sin(\alpha x)}{x}$, $\hat{v}(x) = \frac{2\sin(\beta x)}{x}$

forts →

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{\sin(\alpha x)\sin(\beta x)}{x^2}\,dx = \frac{1}{4}\int \frac{2\sin(\alpha x)}{x}\cdot\overline{\frac{2\sin(\beta x)}{x}} =$$

$$= \frac{2\pi}{4}\int_{-\infty}^{\infty} \chi_\alpha(t)\cdot\overline{\chi_\beta(t)}\,dt = \frac{\pi}{2}\int_{-\min(\alpha,\beta)}^{\min(\alpha,\beta)} 1\,dt =$$

$$= \frac{\pi}{\cancel{2}}\cdot\cancel{2}\min(\alpha,\beta). \blacksquare$$

5.2.5

Beräkna $I = \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2}$

Lösn: Låt $u(t) = \frac{1}{1+t^2}$. Då $\hat{u}(x) = \pi e^{-|x|}$

$$\Longrightarrow I = \int_{-\infty}^{\infty} |u(t)|^2\,dt = \{\text{Parseval}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(x)|^2\,dx$$

$$= \frac{1}{2\cancel{\pi}}\int_{-\infty}^{\infty} \pi^{\cancel{2}} e^{-2|x|}\,dx = \{\text{jämn}\} = \frac{\pi}{\cancel{2}}\cdot\cancel{2}\int_0^{\infty} e^{-2x}\,dx =$$

$$= \pi\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = \frac{\pi}{2}$$

5.3 Laplacetransformer

Definition

Antag $u: \mathbb{R} \to \mathbb{C}$ s.a

(i) $u(t) = 0$ då $t < 0$

(ii) $|u(t)| \leq Me^{at}$ då $t > 0$, $M, a > 0$

Låt $\tilde{u}(s) = (\mathcal{L}u)(s) = \int_0^{\infty} u(t)e^{-st}\,dt$, $s \in \mathbb{C}$

$\tilde{u}, \mathcal{L}u$ kallas Laplacetransformen av u.

Väldefinierad om $\mathrm{Re}(s) > a$.
Kommer alltid anta att $u(t) = 0$ då $t < 0$.
Finns omfattande tabell på hemsidan/tentan!
Laplacetransf. betydligt jobbigare än Fouriertr. teoretiskt, kommer fr.o.m nu att strunta i att hålla koll på när operationer är tillåtna.

5.3.13

Finn $u(t)$ om $(\mathcal{L}u)(s) = \dfrac{s}{(s^2+A^2)^2}$

lösn: Vet från tabellen (nr 14) att

$$v(t) = \sin(At) \Longrightarrow (\mathcal{L}v)(s) = \frac{A}{s^2+A^2}$$

$$\Longleftrightarrow \int_0^\infty \sin(At) e^{-st}\,dt = \frac{A}{s^2+A^2}$$

Derivera båda sidor m.a.p s:

$$\cancel{-}\int_0^\infty \sin(At)\, t\, e^{-st}\,dt = \cancel{-}\frac{2sA}{(s^2+A^2)^2}$$

$$\Longleftrightarrow \frac{s}{(s^2+A^2)^2} = \int_0^\infty \frac{t\sin(At)}{2A} e^{-st}\,dt$$

$\therefore\ u(t) = \dfrac{t\sin(At)}{2A}$ (då $t>0$, $u(t)=0$ då $t<0$)

5.4.2

Lös ODE:n $\quad u''(t) + 9u(t) = \begin{cases} 1 & 0<t<\pi \\ 0 & t>\pi \end{cases}$

$u(0)=1$, $u'(0)=1$

m.hj.a Laplacetr.

lösn: Låt $f(t) =$ HL. Vi har att

$(\mathcal{L}u)(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\pi 1\cdot e^{-st}dt = \dots =$

$= \frac{1}{s}(1-e^{-\pi s})$

$\mathcal{L}(u''+9u) = \mathcal{L}(u'') + 9\hat{u}$

Tabell (nr. 6) : $\mathcal{L}(u'')(s) = s^2\hat{u}(s) - su(0) - u'(0) = s^2\hat{u}(s) - s$

$\Rightarrow s^2\hat{u}(s) - s + 9\hat{u}(s) = \frac{1}{s} - \frac{1}{s}e^{-\pi s}$

$\Leftrightarrow \hat{u}(s) = \frac{s}{s^2+9} +$

$+ \frac{1}{s(s^2+9)} - \frac{1}{s(s^2+9)}e^{-\pi s}$

Tabell (nr. 15) : $\mathcal{L}(\cos(ct))(s) = \frac{s}{s^2+c^2}$

Tabell (nr. 12) : $\mathcal{L}(1)(s) = 1/s$

Tabell (nr. 14) : $\mathcal{L}(\sin(ct))(s) = \frac{c}{s^2+c^2}$

Om $(u * v)(t) = \int_{-\infty}^{\infty} u(x)v(t-x)\,dx = \int_0^t u(x)(v(t-x))\,dx$

så tabell (nr. 10) : $\mathcal{L}(u*v)(s) = \hat{u}(s)\hat{v}(s)$

Tabell (nr. 2) : $\mathcal{L}(H(t-a)u(t-a))(s) = e^{-as}\hat{u}(s)$

där $H(t) = \begin{cases} 1 & \text{om } t>0 \\ 0 & \text{om } t<0 \end{cases}$

Låt $g(t) = 1$, $h(t) = \sin(3t)$. Då

$\mathcal{L}(g*h)(s) = \frac{1}{s}\frac{3}{s^2+9} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s(s^2+9)}\right) = \frac{1}{3}(g*h)(t) =$

$= \frac{1}{3}\int_0^t g(x)h(t-x)\,dx = \frac{1}{3}\int_0^t \sin(3t-3x)\,dx =$

$= \frac{1}{3}\left[\frac{\cancel{-}\cos(3t-3x)}{\cancel{-}3}\right]_{x=0}^{x=t} = \frac{1}{9}(1-\cos(3t))$

$$\because\ u(t) = \cos(3t) + \frac{1}{9} - \frac{\cos(3t)}{9} - H(t-\pi)\left(\frac{1}{9} - \frac{\cos(3(t-\pi))}{9}\right)$$

$$= \frac{8\cos(3t)}{9} + \frac{1}{9} - H(t-\pi)\left(\frac{1}{9} + \frac{\cos 3t}{9}\right) =$$

$$\left\{ \begin{aligned} &\cos(3t-3\pi) = \cos(3t+\pi) = \cos(3t)\underbrace{\cos\pi}_{=-1} - \sin(3t)\underbrace{\sin\pi}_{=0} = \\ &= -\cos(3t). \end{aligned} \right.$$

$$\therefore \begin{cases} \dfrac{8\cos(3t)}{9} + \dfrac{1}{9} & 0 < t < \pi \\[2ex] \dfrac{7\cos(3t)}{9} & t > \pi \end{cases}$$