MVE095 Theorems & Proofs

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Notes

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1 Definitions

Definition 1.1

At time $t < T$ is called an optimal exercise time for the American put with value $\hat{P}(t, S(t), K, T)$ if

$$\hat{P}(t, S(t), K, T) = (K - S(t))_+ \,.$$

Definition 2.2

A portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ invested in a binomial market is said to be self-financing if

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1)$$

holds for all $t \in \mathcal{I}$.

Definition 2.3

A portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ invested in a binomial market is called an arbitrage portfolio if its value $V(t)$ satisfies

- $V(0) = 0$,
- $V(N, x) \geq 0 \forall\, x \in \{u, d\}^N$,
- There exists $y \in \{u, d\}^N$ such that $V(N, y) > 0$.

Definition 3.1

A portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is called **predictable** if there exists N functions $H_1, ..., H_N$ such that $H_t : (0, \infty)^t \to \mathbb{R}^2$ and

$$(h_S(t), h_B(t)) = H_t(S_0, ..., S(t-1)), \qquad t \in \mathcal{I}\,.$$

Definition 3.2

A **hedging** portfolio for a European derivative with pay-off $Y = g(S(N))$ at expiration date $T = N$ is a portfolio process $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ invested in the underlying stock and risk-free asset such that its value $V(t)$ satisfies $V(N) = Y$; the latter equality must be satisfied for all possible paths of the price of the underlying stock, i.e., $V(N,x) = Y(x)\forall x \in \{u, d\}^N$.

Definition 3.3

The binomial (fair) price of a European derivative with pay-off Y and maturity N is given by

$$\Pi_Y(t) := e^{-r(N-T)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1, \ldots, x_N)\,.$$

Definition 4.1

A portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is said to be hedging an American derivative with intrisic value $Y(t)$ if

$$V(N) = Y(N)\,, \qquad V(t) \geq Y(t)\forall t = 0, \ldots, N-1\,,$$

where $V(t) = h_S S(t) + h_B B(t)$ is the value of the portfolio process at time t.

Definition 4.2

The binomial (fair) price $\hat{\Pi}_Y(t)$ of a standard American derivative with pay-off $Y(t) = g(S(t))$ at time $t \in \{0, 1, \ldots, N\}$ is defined by the recurrence formula

$$\begin{array}{l} \hat{\Pi}_Y(N) = Y(N) \\ \hat{\Pi}_Y(t) = \max(Y(t), e^{-r}(q_u\hat{\Pi}^u_Y(t+1) + q_d\hat{\Pi}^d_Y(t+1))) \end{array}\,.$$

Definition 4.3

A replicating portfolio process for an American derivative with intrinsic value $Y(t)$ is a portfolio process that satisfies $V(t) = \hat{\Pi}_Y(t)$, for all $t \in \{0, \ldots, N\}$ (and for all possible paths of the stock price).

Definition 4.4

A portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is said to generate cash flow $C(t-1)$, $t \in \mathcal{I}$, if

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1) - C(t-1)\,, t \in \mathcal{I}\,,$$

or, equivalently

$$V(t) - V(t-1) = h_S(t)(S(t) - S(t-1)) + h_B(t)(B(t) - B(t-1)) - C(t-1)\,.$$

Definition 5.4

Two events A and B are said to be independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Definition 5.15

A discrete stochastic process $\{X_1, X_2, ...\}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a martingale if

$$\mathbb{E}[X_{i+1}|X_1, X_2, ..., X_i] = X_i \, \forall i \geq 1 \, .$$

Definition 5.19

Let $\{W(t)\}_{t\in[0,T]}$ be a Brownian motion, $\alpha \in \mathbb{R}$, and $\sigma > 0$. The positive stochastic process $\{S(t)\}_{t\in[0,T]}$

$$S(t) = S(0)e^{\alpha t + \sigma W(t)} \, ,$$

is called a geometric Brownian motion.

Definition 6.1

Consider a European derivative with pay-off $Y = g(S(T))$ at the maturity $T > 0$. Assume that the price of the underlying stock is given by the geometric brownian motion $S(t) = S(0)e^{\alpha t+\sigma W(t)}$. The Black-Scholes price $\Pi_Y(t)$ of the derivative at time $t \in [0, T]$ is $\Pi_Y(t) = v(t, S(t)$ where

$$v(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}} \, \mathrm{d}y \, , \qquad \tau = T - t \, .$$

2 Theorems

Theorem 1.1

Let $C(t, S(t), T, K)$ denote the price of a European call, and let $P(t, S(t), T, K)$ be the price of the corresponding European put. Assume that there exists a risk-free asset in the money market with constant interest rate r. If the dominance principle holds, then for all $t < T$,

1. The **put-call parity** holds

$$S(t) - C(t, S(t), T, K) = Ke^{-r(T-t)} - P(t, S(t), T, K) \, .$$

2. If $r \geq 0$ then $C(t, S(t), T, K) \geq (S(t) - K)_+$; the strict inequality holds for $r > 0$.
3. If $r \geq 0$, the map $T \mapsto C(t, S(t), T, K)$ is non-decreasing.
4. The maps $K \mapsto C(t, S(t), T, K)$ and $K \mapsto P(t, S(t), T, K)$ are convex.

Proof

1. Consider a portfolio $\mathcal{A}$ which is long one share of the stock and one share of the put option, and short of the call and $K/B(T)$ shares of the risk-free asset. The value of this portfolio at maturity is

$$V_{\mathcal{A}}(T) = S(T) + (K - S(T))_+ - (S(T) - K)_+ - \frac{K}{B(T)}B(T) = 0 \, .$$

Hence by the dominance principle $V_{\mathcal{A}} \geq 0$ for $t < T$, that is

$$S(t) + P(t, S(t), K, T) - C(t, S(t), K, T) - Ke^{-r(T-T)} \geq 0\,.$$

Now consider the portfolio $-\mathcal{A}$ with the opposite position on each asset. Again we have $V_{-\mathcal{A}}(T) = 0$ and thus $V_{-\mathcal{A}}(t) = -V_{\mathcal{A}}(t) \geq 0$ for $t < T$. Hence

$$S(t) + P(t, S(t), K, T) - C(t, S(t), K, T) - Ke^{-r(T-t)} \leq 0\,.$$

Thus the left hand side in the previous two inequalities must be zero, which gives the put-call parity.

2. We can assume $S(t) \geq K$ otherwise it's trivial. By the put-call parity, using that $P(t, S(t), K, T) \geq 0$,

$$C(t, S(t), K, T) = S(t) - Ke^{-r(T-t)} + P(t, S(t), K, T) \geq S(t) - Ke^{-r(T-t)}\,;$$

the right hand side equals $S(t) - K$ for $r = 0$ and is strictly greater than this quantity for $r > 0$. As $S(t) - K = (S(t) - K)_+$ for $S(t) \geq K$, the claim follows.

3. Consider a portfolio $\mathcal{A}$ which is long one call with maturity T_2 and strike K and short one call with maturity T_1 and strike K, where $T_2 > T_1 \geq t$. By claim 2 we have

$$C(T_1, S(T_1), K, T_2) \geq (S(T_1) - K)_+ = C(t_1, S(T_1), K, T_1)\,,$$

i.e. $V_{\mathcal{A}}(T_1) \geq 0$ for $t < T_1$. Hence $V_{\mathcal{A}}(t) \geq 0$ i.e. $C(t, S(t), K, T_2) \geq C(t, S(t), K, T_1)$, which is the claim.

4. We prove the statement for call options, the argument for put options being the same. Let $K_0, K_1 > 0$ and $0 < \lambda < 1$ be given. Consider a portfolio $\mathcal{A}$ which is short one share with strike $\lambda K_0 + (1-\lambda)K_1$ and maturity T, long λ shares of a call with strike K_1 and maturity T, long $(1-\lambda)$ shares of a call with strike K_0 and maturity T. The value of this portfolio at maturity is

$$V_{\mathcal{A}}(T) = -(S(T) - (\lambda K_1 + (1-\lambda)K_0))_+ + \lambda(S(T) - K_1)_+ + (1-\lambda)(S(T) - K_0)_+\,.$$

The convexity of the function $f(x) = (S(T) - x)_+$ gives $V_{\mathcal{A}}(T) \geq 0$, and so $V_{\mathcal{A}}(t) \geq 0$ by the dominance principle. The latter inequality is

$$C(t, S(t), \lambda K_1 + (1-\lambda)K_0, T) \leq \lambda C(t, S(t), K_1, T) + (1-\lambda)C(t, S(t), K_0, T)\,,$$

which is the claim for call options.

Theorem 2.1

Let $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ be a self-financing portfolio process with value $V(N)$ at time $t = N$. Define

$$q_u = \frac{e^r - e^d}{e^u - e^d}\,, \qquad q_d = 1 - q_u\,.$$

Then for $t = 0, ..., N-1$, $V(t)$ is given by

$$V(t) = e^{-r(N-T)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x)\,.$$

In particular we have the initial value

$$V(0) = e^{-rN} \sum_{x \in \{u,d\}^N} q_u^{N_u(x)} q_d^{N_d(x)} V(N, x) \,.$$

Moreover the portfolio satisfies the recurrence formula

$$V(t-1) = e^{-r}[q_u V^u(t) + q_d V^d(t)] \,, \qquad t \in \mathcal{I} \,,$$

where

$$V^u(t) = h_S(t)S(t-1)e^u + h_B(t)B(t-1)e^r \,,$$
$$V^d(t) = h_S(t)S(t-1)e^d + h_B(t)B(t-1)e^d \,.$$

Proof

We prove it by induction for $t = 0, ..., N-1$.

Step 1

We begin with $t = N-1$. Then

$$V(N-1) = e^{-r}[q_u V(N, (x_1, ..., x_{N-1}, u)) + q_d V(N, (x_1, ..., x_{N-1}, d))] \,. \tag{1}$$

Here, we have

$$V(N, (x_1, ..., x_{N-1}, u)) = h_S(N)S(N-1)e^u + h_B(N)B(N-1)e^r \,,$$

similarily for $V(N, (x_1, ..., x_{N-1}, d))$ but u replaced with d, which follows by the definition of portfolio vlaue. Thus $V(N-1)$ is equal to

$$\begin{aligned}
&V(N-1) \\
&= e^{-r}[q_u(h_S(N)S(N-1)e^u + h_B(N)B(N-1)e^r) \\
&+q_d(h_S(N)S(N-1)e^d + h_B(N)B(N-1)e^r)] \\
&= e^{-r}[h_S(N)S(N-1)e^r + h_B(N)B(N-1)e^r] \\
&= h_S(N)S(N-1) + h_B(N)B(N-1) \,,
\end{aligned}$$

since $e^u q_u + e^d q_d = e^r$ and $q_u + q_d = 1$. This proves the claim for $t = N-1$, by the definition of self-financing portfolios.

Step 2

Now assume this is true at $t+1$ i.e.

$$V(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, ..., x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x) \,. \tag{2}$$

Step 3

We now prove it at time t. Let

$$\begin{aligned} V^u(t+1) &:= h_S(t+1)S(t)e^u + h_B(t+1)B(t)e^r \quad \text{assuming } x_{t+1} = u\,, \\ V^d(t+1) &:= h_S(t+1)S(t)e^d + h_B(t+1)B(t)e^r \quad \text{assuming } x_{t+1} = d\,. \end{aligned}$$

This gives us

$$e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)] = h_S(t+1)S(t) + h_B(t+1)B(t)\,.$$

By the self financing property we have

$$e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)] = V(t)\,,$$

which proves that V satisfies the recurrence formula. Moreover, with the induction hypothesis we have

$$V^u(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2},\dots,x_N)\in\{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \dots, x_t, u, x_{t+2}, \dots, x_N)\,,$$

$$V^d(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2},\dots,x_N)\in\{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \dots, x_t, d, x_{t+2}, \dots, x_N)\,,$$

using these, with equation (2), we obtain

$$V(t) = e^{-r(N-T)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x)\,.$$

Theorem 2.2

The binomial market is arbitrage free iff $r \in (d, u)$.

Proof

The proof is divided into 2 steps, first we prove the claim for the 1-period model. The generalization for the multiperiod model $N > 1$ is carried out in step 2.

Step 1

Let the portfolio position in the 1-period model be constant, (thus be self-financing over $[0, 1]$), i.e. let

$$h_S(0) = h_S(1) = h_S\,, \qquad h_B(0) = h_B(1) = h_B\,. \tag{3}$$

The portfolio value at $t = 0$ is

$$V(0) = h_S S_0 + h_B B_0\,, \tag{4}$$

while at time $t = 1$ it is

$$V(1) = \begin{cases} V(1, u) = h_S S_0 e^u + h_B B_0 e^r & \text{if stock goes up at } t = 1 \\ V(1, d) = h_S S_0 e^d + h_B B_0 e^r & \text{if stock goes down at } t = 1 \end{cases} \tag{5}$$

Thus the portfolio is an arbitrage if $V(0) = 0$, i.e.

$$h_s S_0 + h_B B_0 = 0\,, \tag{6}$$

if $V(1) \geq 0$ i.e.

$$\begin{aligned} h_S S_0 e^u + h_B B_0 e^r \geq 0 \\ h_S S_0 e^d + h_B B_0 e^r \geq 0 \end{aligned} \tag{7}$$

and if at least one of the two inequalities in (7) is strict. Now assume that (h_S, h_B) is an arbitrage portfolio. From (6) we have $h_S S_0 = -h_B B_0$, thus (7) becomes

$$h_S S_0(e^u - e^r) \geq 0\,, \tag{8}$$

$$h_S S_0(e^d - e^r) \geq 0\,. \tag{9}$$

We have $h_S \neq 0$ since at least one of the inequalities must be strict. Assuming $h_S > 0$ then we obtain from the two inequalities above that $d \geq r$. Instead, assuming $h_S < 0$ we instead obtain $r \geq u$. Hence the existence of an arbitrage portfolio implies $r \geq u$ or $r \leq d$, i.e. $r \notin (d, u)$. Which proves that for $r \in (d, u)$ there is no arbitrage portfolio for the 1-period model. Now we need to prove that $r \in (d, u)$ is necessary for the absence of arbitrages, we construct an arbitrage portfolio when $r \notin (d, u)$. Assume $r \leq d$, pick $h_S = 1$ and $h_B = -S_0/B_0$. Then $V(0) = 0$. Further, $h_S S_0 e^d + h_B B_0 e^r \geq 0$ is trivially satisfied, and since $u > d$ we have

$$h_S S_0 e^u + h_B B_0 e^r = S_0(e^u - e^r) > S_0(e^d - e^r) \geq 0\,. \tag{10}$$

This shows that one can construct an arbitrage portfolio when $r \leq d$, a similiar argument is done for $r \geq u$. We now continue with step 2.

Step 2

Again let $r \notin (d, u)$, we've shown that in the 1-period model there exists an arbitrage portfolio (h_S, h_B). Now by investing the whole value of the portfolio (h_S, h_B) at $t = 1$ in the risk-free asset, we can build a self-financing arbitrage portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ for the multiperiod model. This portfolio satisfies $V(0) = 0$ and $V(N, x) = V(1, x)e^{r(N-1)} \geq 0$ along every path $x \in \{u, d\}^N$. Moreover, (h_S, h_B) is an arbitrage, therefore $V(1, y) > 0$ for some $y \in \{u, d\}^N$, hence $V(N, y) > 0$. The constructed self-financing portfolio process $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is an arbitrage, now we have to prove the "only if" part for the multiperiod model. By Theorem 2.1

$$V(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(N, x)\,. \tag{11}$$

Now assume that the portfolio is an arbitrage. Then $V(0) = 0$ and $V(N, x) \geq 0$. We can consider only paths such that $V(N, x) > 0$ which exists since the portfolio is an arbitrage. But then (11) can be zero only if one of q_u or q_d is zero, or if opposite signs. Since $u > d$ we have

$$\begin{aligned} &q_u = 0\,, \qquad \text{resp. } q_d = 0 \Rightarrow r = d\,, \qquad \text{resp. } u = r \\ &(q_u > 0\,, q_d < 0)\,, \qquad \text{resp. } (q_u < 0\,, q_d > 0) \Rightarrow u < r\,, \qquad \text{resp. } r < d\,. \end{aligned} \tag{12}$$

We conclude that the existence of a self-financing arbitrage portfolio entails $r \notin (d, u)$ which completes the proof.

Theorem 3.2

Consider a standard European derivative with pay-off $Y = g(S(N))$ at the time of maturity N. Then the portfolio given by

$$h_S(0) = h_S(1)\,, \qquad h_B(0) = h_B(1)\,,$$

and for $t \in \mathcal{I}$,

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}\,,$$

$$h_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d}\,,$$

is a self-financing, predictable, hedging portfolio.

Proof

We begin proving the hedging property, we have

$$V(t) = h_S(t)S(t) + h_B(t)B(t) = \frac{S(t)}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} + \frac{e^{-r}B(t)}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d}\,.$$

Here $e^{-r}B(t)/B(t-1) = 1$ and $S(t)/S(t-1)$ is either e^u or e^d depending on $S(t)$. Using these two values we obtain $V_Y^u(t) = \Pi_Y^u(t)$, and $V_Y^d(t) = \Pi_Y^d(t)$, that is $V(t) = \Pi_Y(t)$ i.e. replicating the derivative. In particular, for $t = N$ we have $V(N) = \Pi_Y(N) = Y$, hence the portfolio is hedging the derivative.
Now, proving the self-financing property, we have

$$h_S(t)S(t-1) + h_B(t)B(t-1) = \frac{\Pi_Y^u(t)(1 - e^{d-r}) + \Pi_Y^d(t)(e^{u-r} - 1)}{e^u - e^d} = \Pi_Y(t-1)\,,$$

by using the definition of q_u, q_d as well as the recurrence formula. Also we already know that the portfolio is replicating the derivative, i.e. $V(t-1) = \Pi_Y(t-1)$, therefore

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1)\,.$$

Finally, the portfolio is predictable, since

$$\Pi_Y(t) \coloneqq e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t)\exp(x_{t+1} + \ldots + x_N))\,,$$

therefore, we have

$$\Pi_Y^u(t) \coloneqq e^{-r(N-t)} \sum_{(x_{t+1},\ldots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t-1)e^u \exp(x_{t+1} + \ldots + x_N))\,,$$

hence $\Pi_Y^u(t)$ is a deterministic function of $S(t-1)$ and the same property holds for $\Pi_Y^d(t)$. Thus $h_S(t), h_B(t)$ are deterministic functions of $S(t-1)$, which proves that the portfolio is predictable.

Theorem 4.1

Consider a standard American derivative with intrinsic value $Y(t)$ and let $\hat{\Pi}_Y(t)$ be its binomial fair price. Define the portfolio process $\{\hat{h}_S(t), \hat{h}_B(t)\}_{t\in\mathcal{I}}$ and the cash flow process $C(t)$ recursively as follows:

$$C(0) = 0\,, \qquad C(t-1) = \hat{\Pi}_Y(t-1) - e^{-r}[q_u\hat{\Pi}_Y^u(t) + q_d\hat{\Pi}_Y^d(t)]\,, \qquad t \in \{2, ..., N\} \tag{13}$$

$$\hat{h}_S(1) = \hat{h}_S(0)\,, \qquad \hat{h}_B(1) = \hat{h}_B(0)\,, \tag{14}$$

and for $t \in \{1, ..., N\}$,

$$\hat{h}_S(t) = \frac{1}{S(t-1)}\frac{\hat{\Pi}_Y^u(t) - \hat{\Pi}_Y^d(t)}{e^u - e^d}\,, \tag{15}$$

$$\hat{h}_B(t) = \frac{e^{-r}}{B(t-1)}\frac{e^u\hat{\Pi}_Y^d(t) - e^d\hat{\Pi}_Y^u(t)}{e^u - e^d}\,. \tag{16}$$

Then the value of this portfolio process satisfies

$$V(t) = \hat{\Pi}_Y(t)\,\forall t \in \{0, ..., N\}\,, \tag{17}$$

and

$$V(t-1) = \hat{h}_S(t)S(t-1) + \hat{h}_B(t)B(t-1) + C(t-1)\,, \qquad \forall t \in \mathcal{I}\,. \tag{18}$$

Proof

By using the equations (13), (14), (15), (16) into equations (17) and (18) we obtain

$$V^u(t) = \hat{h}_S(t)S(t-1)e^u + \hat{h}_B(t)B(t-1)e^d = \hat{\Pi}_Y^u(t)\,,$$

similiar calculations proves $V^d(t) = \hat{\Pi}_Y^d(t)$, hence (17) holds. Also replacing equations (13), (14), (15), (16) into the right hand side of (18), the latter is equal to $\hat{\Pi}_Y(t-1)$, which we proved is equal to $V(t-1)$, hence (18) holds.

Theorem 5.3

If $r \notin (d, u)$ there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price $\{\hat{S}(t)\}_{t\in\mathcal{I}}$ is a martingale. For $r \in (d, u)$, $\{\hat{S}(t)\}_{t\in\mathcal{I}}$ is a martingale with respect to the probability measure $\mathbb{P}_q$ where

$$q = \frac{e^r - e^d}{e^u - e^d}\,.$$

Moreover $\mathbb{P}_q$ is the only probability meausure on Ω_N for which $\{\hat{S}(t)\}_{t\in\mathcal{I}}$ is a martingale.

Proof

By definition $\{\hat{S}(t)\}_{t\in\mathcal{I}}$ is a martingale if and only if

$$\mathbb{E}[e^{-rt}S(t)|\hat{S}(1), ..., \hat{S}(t-1)] = e^{-r(t-1)}S(t-1)\,, \forall t \in \mathcal{I}. \tag{19}$$

Clearly, conditioning on $\hat{S}(1), ..., \hat{S}(t-1)$ is the same as taking the expectation conditional to $S(1), ..., S(t-1)$, hence (19) is equivalent to

$$\mathbb{E}[S(t)|S(1), ..., S(t-1)] = e^r S(t-1)\,, \forall t \in \mathcal{I}.$$

Moreover

$$\mathbb{E}[S(t)|S(1), ..., S(t-1)] = \mathbb{E}[S(t-1)\frac{S(t)}{S(t-1)}|S(1), ..., S(t-1)]\,, \tag{20}$$

however since $S(t-1)$ is known, and because $S(t)/S(t-1)$ is either e^u with probability p, or e^d with probability $1-p$, and is independent of $S(1), ..., S(t-1)$, it follows that

$$\mathbb{E}[S(t)|S(1), ..., S(t-1)] = S(t-1)(e^u p + e^d(1-p))\,.$$

Therefore $\mathbb{E}[S(t)|S(1), ..., S(t-1)] = e^r S(t-1)$ holds iff $e^r = e^u p + e^d(1-p)$. The latter has a solution $p \in (0,1)$ iff $r \in (d,u)$, when it exists it is given by $p = q$.

Theorem 5.4

Let $\mathbb{E}_p[\cdot]$ denote the expectation in the probability measure $\mathbb{P}_p$. We have

$$\mathbb{E}_p[S(N)] = S(0)(e^u p + e^d(1-p))^N\,, \qquad \mathbb{E}_q[S(N)] = S(0)e^{rN}\,. \tag{21}$$

Proof

We prove only the first formula because the second formula follows by the first one using that $e^u q + e^d(1-q) = e^r$. We have

$$\mathbb{E}_p[S(N)] = \mathbb{E}_p[S(0)\exp(X_1 + ... + X_N)] = S(0)\mathbb{E}_p[Y]\,, \tag{22}$$

where Y is the random variable $Y = \exp(X_1 + ... + X_2) = \exp(uN_H(\omega) + dN_T(\omega))$, $\omega \in \Omega$. Now using that $N_T = N - N_H$ it follows that

$$\mathbb{E}_p[S(N)] = S(0)\sum_{\omega\in\Omega_N} e^{uN_H + dN_T} p^{N_H}(1-p)^{N_T} = S(0)e^{dN}(1-p)^N \sum_{\omega\in\Omega_N}\left(\frac{e^u p}{e^d(1-p)}\right)^{N_H}\,.$$

Now, since for $k = 0, ..., N$ there is $\binom{N}{k}$ sample points $\omega \in \Omega_N$ such that $N_H(\omega) = k$, we rewrite the above expresssion and using the binomial theorem, we obtain the following

$$\begin{aligned}\mathbb{E}_p[S(N)] = S(0)e^{Nd}(1-p)^N \sum_{k=0}^{N}\binom{N}{k}\left(\frac{e^u p}{e^d(1-p)}\right)^k &= S(0)e^{Nd}(1-p)^N\left(1 + \frac{e^u p}{e^d(1-p)}\right)^N \\ &= S(0)(e^d(1-p)e^u p)^N\,.\end{aligned}$$

Theorem 5.10

The density of the random variable $S(t)$ is given by

$$f_{S(t)}(x) = \frac{\mathbb{I}_{x>0}}{x\sigma\sqrt{2\pi t}}\exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right)\,,$$

where $\mathbb{I}_{x>0}$ is the indicator function of the set $x > 0$.

Proof

The density of $S(t)$ is given by

$$f_{S(t)}(x)\frac{\mathrm{d}}{\mathrm{d}x}F_{S(t)}(x)\,,$$

where $F_{S(t)}$ is the distribution of $S(t)$, i.e.,

$$F_{S(t)}(x) = \mathbb{P}(S(t) \leq x)\,.$$

Clearly, $f_{S(t)} = F_{S(t)} = 0$ for $x \leq 0$. For $x > 0$ we use that

$$S(t) \leq x \qquad \text{if and only if} \qquad W(t) \leq \frac{1}{\sigma}\left(\log\frac{x}{S(0)} - \alpha t\right) \coloneqq A(x)\,.$$

Thus

$$\mathbb{P}(S(t) \leq x) = \mathbb{P}(-\infty < W(t) \leq A(x)) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\,\mathrm{d}t\,,$$

where for the second equality we used that $W(t) \in \mathcal{N}(0,t)$. Hence

$$f_{S(t)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\,\mathrm{d}t\right) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{A(x)^2}{2t}}\frac{\mathrm{d}A(x)}{\mathrm{d}x}$$

for $x > 0$, that is

$$f_{S(t)}(x) = \frac{1}{\sigma x\sqrt{2\pi t}}\exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right)\,, \quad x > 0\,.$$

Theorem 6.2

The Black-Scholes price at time t of a European call option with strike $K > 0$, maturity time $T > 0$ is given by $C(t, S(t))$ where

$$C(t,x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\,,\, d_2 = \frac{\log\left(\frac{x}{K}\right) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\,,\, d_1 = d_2 + \sigma\sqrt{\tau}\,, \tag{23}$$

where Φ is the standard normal distribution. The Black-Scholes price of the corresponding put option is given by $P(t, S(t))$ where

$$P(t,x) = -x\Phi(-d_1) + Ke^{-r\tau}\Phi(-d_2)\,.$$

Moreover the put-call parity holds

$$C(t,S(t)) - P(t,S(t)) = S(t) - Ke^{-r\tau}\,. \tag{24}$$

Proof

We derive the price for call options, since the argument is similiar for put options. Recall the pay-off function $g(z) = (z-K)_+$, we have

$$C(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{\tau(r-\frac{\sigma^2}{2})+\sigma\sqrt{\tau}y} - K\right) e^{-\frac{y^2}{2}}\,\mathrm{d}y\,.$$

Note that g is nonzero when $xe^{\tau(r-\frac{\sigma^2}{2})+\sigma\sqrt{\tau}y} - K$ i.e. when $y > -d_2$. Thus using $-\frac{1}{2}y^2 + \sigma\sqrt{\tau}y = -\frac{1}{2}(y-\sigma\sqrt{\tau})^2 + \frac{\sigma^2\tau}{2}$. Thus we obtain

$$C(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left(xe^{\tau(r-\frac{\sigma^2}{2})+\frac{\sigma^2\tau}{2}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\,\mathrm{d}y - K\int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\,\mathrm{d}y\right)\,.$$

Now consider the left integral with the change of variables $u = y - \sigma\sqrt{\tau}$, which gives the lower integral limit $u = -d_2 - \sigma\sqrt{\tau} = -d_1$. Now since we have two integrals of even functions, symmetric around zero, we have

$$C(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left(xe^{\tau r} \int_{-d_1}^{\infty} e^{-\frac{1}{2}u^2}\,\mathrm{d}u - K\int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\,\mathrm{d}y\right) =$$

$$= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left(xe^{\tau r} \int_{-\infty}^{d_1} e^{-\frac{1}{2}u^2}\,\mathrm{d}u - K\int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\,\mathrm{d}y\right) =$$

$$= x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\,.$$

Finally, the put-call parity follows since

$$C(t,x) - P(t,x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) - (-x\Phi(-d_1) + Ke^{-r\tau}\Phi(-d_2)) =$$

$$= x(\Phi(d_1) + \Phi(d_1)) - Ke^{-r\tau}(\Phi(d_2) + \Phi(d_2)) = x - Ke^{-r\tau}\,,$$

since $\Phi(u) + \Phi(-u) = 1$. Thus the claims follows.