SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

August 29, 2009, morning (4 hours), v
No aids.
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Each problem is worth 3 points.

1. Find a portfolio consisting of European calls and puts with termination date T such that the value of the portfolio at time T equals

$$Y = \min(K, \mid S(T) - K \mid).$$

Solution. By drawing a graph of Y as a function $S(T)$ we get $Y = (K - S(T))^+ + (S(T) - K)^+ - (S(T) - 2K)^+$. Thus a portfolio with long one European put with strike K and expiry T, long one European call with strike K and expiry T, and short one call with strike $2K$ and expiry T will satisfy the requirements in the text.

2. The Black-Scholes call price equals $c = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $\tau = T - t > 0$ and $d_1 = (\sigma\sqrt{\tau})^{-1}\left\{\ln(s/K) + (r + \sigma^2/2)\tau\right\} = d_2 + \sigma\sqrt{\tau}$. Show that

$$\frac{\partial c}{\partial K} = -e^{-r\tau}\Phi(d_2).$$

Solution. Let $\varphi = \Phi'$. We have

$$\frac{\partial c}{\partial K} = s\varphi(d_1)\frac{\partial d_1}{\partial K} - e^{-r\tau}\Phi(d_2) - Ke^{-r\tau}\varphi(d_2)\frac{\partial d_2}{\partial K}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{\partial d_1}{\partial K}\left\{s\varphi(d_1) - Ke^{-r\tau}\varphi(d_2)\right\}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial K}\left\{se^{-d_1^2/2} - Ke^{-r\tau}e^{-d_2^2/2}\right\}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial K}\left\{se^{-d_1^2/2} - Ke^{-r\tau}e^{-d_1^2/2+d_1\sigma\sqrt{\tau}-\sigma^2\tau/2}\right\}$$

$$= -e^{-r\tau}\Phi(d_2) + \frac{Ke^{-d_1^2/2}}{\sqrt{2\pi}}\frac{\partial d_1}{\partial K}\left\{s/K - e^{d_1\sigma\sqrt{\tau}-(r+\sigma^2/2)\tau}\right\} = -e^{-r\tau}\Phi(d_2).$$

3. Let a be a positive real number and suppose the function $u(t, s)$ satisfies the Black-Scholes differential equation

$$u_t' + \frac{\sigma^2 s^2}{2}u_{ss}'' + rsu_s' - ru = 0,\ \ 0 \le t < T,\ \ s > 0.$$

Show that the function $v(t, s) = s^{1-\frac{2r}{\sigma^2}}u(t, \frac{a}{s})$ satisfies the Black-Scholes differential equation.

Solution. We have

$$v_t'(t, s) = s^{1-\frac{2r}{\sigma^2}}u_t'(t, \frac{a}{s})$$

$$v_s'(t, s) = (1 - \frac{2r}{\sigma^2})s^{-\frac{2r}{\sigma^2}}u(t, \frac{a}{s}) - as^{-1-\frac{2r}{\sigma^2}}u_s'(t, \frac{a}{s})$$

and

$$v_{ss}''(t, s) = -\frac{2r}{\sigma^2}(1 - \frac{2r}{\sigma^2})s^{-1-\frac{2r}{\sigma^2}}u(t, \frac{a}{s}) - a(1 - \frac{2r}{\sigma^2})s^{-2-\frac{2r}{\sigma^2}}u_s'(t, \frac{a}{s})$$

$$+a(1 + \frac{2r}{\sigma^2})s^{-2-\frac{2r}{\sigma^2}}u_s'(t, \frac{a}{s}) + a^2 s^{-3-\frac{2r}{\sigma^2}}u_{ss}''(t, \frac{a}{s})$$

$$= -\frac{2r}{\sigma^2}(1 - \frac{2r}{\sigma^2})s^{-1-\frac{2r}{\sigma^2}}u(t, \frac{a}{s}) + a\frac{4r}{\sigma^2}s^{-2-\frac{2r}{\sigma^2}}u_s'(t, \frac{a}{s}) + a^2 s^{-3-\frac{2r}{\sigma^2}}u_{ss}''(t, \frac{a}{s}).$$

Thus

$$v_t' + \frac{\sigma^2 s^2}{2}v_{ss}'' + rsv_s' - rv$$

$$= s^{1-\frac{2r}{\sigma^2}}(u_t'(t, \frac{a}{s}) - r(1 - \frac{2r}{\sigma^2})u(t, \frac{a}{s}) + a2rs^{-1}u_s'(t, \frac{a}{s}) + a^2\frac{\sigma^2}{2}s^{-2}u_{ss}''(t, \frac{a}{s})$$

$$+r(1 - \frac{2r}{\sigma^2})u(t, \frac{a}{s}) - ars^{-1}u_s'(t, \frac{a}{s}) - ru(t, \frac{a}{s}))$$

$$= s^{1-\frac{2r}{\sigma^2}}(u_t'(t, \frac{a}{s}) + \frac{\sigma^2}{2}(\frac{a}{s})^2 u_{ss}''(t, \frac{a}{s}) + r\frac{a}{s}u_s'(t, \frac{a}{s}) - ru(t, \frac{a}{s})) = 0.$$

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P\left[X_1 = 1\right] = P\left[X_1 = -1\right] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. (Dominance principle) Show that the map

$$K \to c(t, S(t), K, T),\ K > 0$$

is convex.