SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

August 27, 2011, morning, v.
No aids.
Each problem is worth 3 points.
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1. (Binomial model with $T = 2$, $u > 0$, $d = -u$, and $0 < r < u$) A financial derivative of European type pays the amount Y at time of maturity T, where

$$Y = S(1) - \min_{t \in \{1,2\}} S(t).$$

Find $\Pi_Y(0)$.

Solution. Set $S(0) = s$ and

$$S(t+1) = S(t)e^{X_{t+1}},\ t = 0, 1.$$

If $v(t) = \Pi_Y(t)$,

$$\begin{aligned}
v(2)_{|X_1=u,X_2=u} &= se^u - se^u = 0 \\
v(2)_{|X_1=u,X_2=d} &= se^u - s = s(e^u - 1) \\
v(2)_{|X_1=d,X_2=u} &= se^{-u} - se^{-u} = 0 \\
v(2)_{|X_1=d,X_2=d} &= se^{-u} - se^{-2u} = se^{-2u}(e^u - 1)
\end{aligned}$$

and defining

$$q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} = 1 - q_d$$

we get

$$\begin{aligned}
v(1)_{|X_1=u} &= e^{-r}(q_u \cdot 0 + q_d s(e^u - 1)) = e^{-r} q_d s(e^u - 1) \\
v(1)_{|X_1=d} &= e^{-r}(q_u \cdot 0 + q_d se^{-2u}(e^u - 1)) = e^{-r} q_d se^{-2u}(e^u - 1).
\end{aligned}$$

Thus

$$v(0) = e^{-r}\left\{q_u e^{-r} q_d s(e^u - 1) + q_d e^{-r} q_d se^{-2u}(e^u - 1)\right\} =$$

$$e^{-2r}sq_d(e^u-1)(q_u+q_de^{-2u}).$$

2. (Binomial model with T periods and $u > r > d$) A financial derivative of European type pays the amount $Y=\sqrt{S(T)}$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. Set

$$q_u=\frac{e^r-e^d}{e^u-e^d} \text{ and } q_d=\frac{e^u-e^r}{e^u-e^d}.$$

We have

$$\Pi_Y(0)=e^{-rT}\sum_{k=0}^{T}\left(\begin{array}{c}T\\k\end{array}\right)q_u^kq_d^{T-k}\sqrt{S(0)e^{ku+(T-k)d}}=$$

$$e^{-rT}\sqrt{S(0)}\sum_{k=0}^{T}\left(\begin{array}{c}T\\k\end{array}\right)(q_ue^{u/2})^k(q_de^{d/2})^{(T-d)}=$$

$$e^{-rT}\sqrt{S(0)}(q_ue^{u/2}+q_de^{d/2})^T.$$

3. (Black-Scholes model) Suppose $0<a<b$ and $0\leq t<T$. A financial derivative of European type pays the amount Y at time of maturity T, where

$$Y=\left\{\begin{array}{c}S(T) \text{ if } S(T)\in\,]a,b[\,,\\ 0 \text{ if } S(T)\notin\,]a,b[\,.\end{array}\right.$$

Find $\Pi_Y(t)$.

Solution. (a) Let $H(x)=$

$$H_0(x)=\left\{\begin{array}{l}1 \text{ if } x>0,\\ 0 \text{ if } x\leq 0\end{array}\right. \text{ and } H_1(x)=\left\{\begin{array}{l}1 \text{ if } x\geq 0,\\ 0 \text{ if } x<0.\end{array}\right.$$

Then

$$Y=S(T)(H_0(S(T)-a)-H_1(S(T)-b)).$$

Moreover, if $s = S(t)$ and $\tau = T - t$,

$$\Pi_{S(T)H_0(S(T)-a)}(t) = e^{-r\tau}\int_{-\infty}^{\infty} se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}x}H_0(se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}x}-a)\varphi(x)dx =$$

$$e^{-r\tau}\int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau)} se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}x}\varphi(x)dx = s\int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau)} e^{-\frac{\sigma^2}{2}\tau-\sigma\sqrt{\tau}x-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} =$$

$$s\int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r-\frac{\sigma^2}{2})\tau)} e^{-\frac{1}{2}(\sigma\sqrt{\tau}+x)^2}\frac{dx}{\sqrt{2\pi}} = s\int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r+\frac{\sigma^2}{2})\tau)} e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} =$$

$$= s\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{a}+(r+\frac{\sigma^2}{2})\tau))$$

and, in a similar way,

$$\Pi_{S(T)H_1(S(T)-b)}(t) = s\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{b}+(r+\frac{\sigma^2}{2})\tau)).$$

Thus

$$\Pi_Y(t) =$$

$$S(t)\left(\Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{a}+(r+\frac{\sigma^2}{2})\tau)) - \Phi(\frac{1}{\sigma\sqrt{\tau}}(\ln\frac{S(t)}{b}+(r+\frac{\sigma^2}{2})\tau))\right).$$

4. (a) State the Chebychev inequality. (b) Suppose $(X_k)_{k=1}^n$ is an i.i.d. with $E\left[X_1^2\right] < \infty$. Use Part (a) to show that

$$P\left[\mid \frac{1}{n}(X_1+...+X_n) - E\left[X_1\right] \mid\geq \varepsilon\right] \leq \frac{\mathrm{Var}(X_1)}{\varepsilon^2 n} \text{ if } \varepsilon > 0.$$

5. (Black-Scholes model) Suppose $\tau = T - t > 0$ and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau).$$

Prove that

$$\frac{\partial c}{\partial s}(t,s,K,T) = \Phi(d_1).$$

(Hint: $c(t,s,K,T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $d_2 = d_1 - \sigma\sqrt{\tau}$)