SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

January 14, 2012, morning, v.
No aids.
Each problem is worth 3 points.
Examiner: Christer Borell, telephone number 0705292322

1. (Black-Scholes model) Let $T > 1$ and $K > 0$ and consider a financial derivative of European type with payoff

$$Y = (\frac{S(T)}{S(T-1)} - K)^+$$

at time of maturity T. Find $\Pi_Y(t)$ if $0 \leq t \leq T-1$.

Solution. Since

$$Y = \frac{1}{S(T-1)}(S(T) - KS(T-1))^+$$

the Black-Scholes call price formula yields

$$\Pi_Y(T-1) = \Phi(\frac{\ln \frac{1}{K} + (r + \frac{\sigma^2}{2})}{\sigma}) - Ke^{-r}\Phi(\frac{\ln \frac{1}{K} + (r - \frac{\sigma^2}{2})}{\sigma}).$$

Thus if $\tau = T - t$ and $0 \leq t \leq T-1$,

$$\Pi_Y(t) = e^{-r(\tau-1)} \left\{ \Phi(\frac{\ln \frac{1}{K} + (r + \frac{\sigma^2}{2})}{\sigma}) - Ke^{-r}\Phi(\frac{\ln \frac{1}{K} + (r - \frac{\sigma^2}{2})}{\sigma}) \right\}.$$

Alternative solution. If $s = S(t)$ and $G \in N(0,1)$,

$$\Pi_Y(t) = e^{-r\tau} E\left[(\frac{se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W(T)-W(t))}}{se^{(r-\frac{\sigma^2}{2})(T-1-t)+\sigma(W(T-1)-W(t))}} - K)^+ \right]$$

$$= e^{-r\tau} E\left[(e^{(r-\frac{\sigma^2}{2})+\sigma(W(T)-W(T-1))} - K)^+ \right]$$

$$= e^{-r(\tau-1)}e^{-r}E\left[(e^{(r-\frac{\sigma^2}{2})+\sigma G} - K)^+\right]$$

$$= e^{-r(\tau-1)}c(0,1,K,1)$$

$$= e^{-r(\tau-1)}\left\{\Phi(\frac{\ln\frac{1}{K} + (r+\frac{\sigma^2}{2})}{\sigma}) - Ke^{-r}\Phi(\frac{\ln\frac{1}{K} + (r-\frac{\sigma^2}{2})}{\sigma})\right\}.$$

2. (Binomial model in T period with $d < r < u$) A financial derivative of European type pays the amount

$$Y = \ln\frac{S(T)}{S(0)}$$

at time of maturity T. Find $\Pi_Y(0)$.

Solution. Using standard notation,

$$Y = \ln\frac{S(T-1)}{S(0)} + X_T$$

and, hence,

$$\Pi_Y(T-1) = e^{-r}\ln\frac{S(T-1)}{S(0)} + e^{-r}(q_u u + q_d d)$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^r} = 1 - q_d.$$

In a similar way, if $T \geq 2$,

$$\Pi_Y(T-2) = e^{-r}(e^{-r}\ln\frac{S(T-2)}{S(0)} + e^{-r}(q_u u + q_d d)) + e^{-2r}(q_u u + q_d d)$$

$$= e^{-2r}\ln\frac{S(T-2)}{S(0)} + 2e^{-2r}(q_u u + q_d d)$$

and by iteration

$$\Pi_Y(0) = Te^{-Tr}(q_u u + q_d d).$$

3. A random variable X has the density

$$f(x) = \begin{cases} \frac{1}{\pi}\sin^2 x \text{ if } \mid x \mid \leq \pi, \\ 0 \text{ otherwise.} \end{cases}$$

Find the characteristic function of X.

Solution. We have that

$$c_X(\xi) = E\left[e^{i\xi X}\right] = \int_{-\pi}^{\pi} e^{i\xi x}\frac{1}{\pi}\sin^2 x dx$$

$$= \int_{-\pi}^{\pi}(\cos\xi x + i\sin\xi x)\frac{1}{\pi}\sin^2 x dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\xi x\sin^2 x dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2x)\cos\xi x dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(1-\cos 2x)\cos\xi x dx = \frac{1}{\pi}(\frac{\sin\pi\xi}{\xi} - a)$$

where

$$a = \int_0^{\pi}\cos 2x\cos\xi x dx$$

$$= \frac{1}{2}\int_0^{\pi}(\cos(2+\xi)x + \cos(2-\xi)x)dx$$

$$= \frac{1}{2}(\frac{\sin(2+\xi)\pi}{2+\xi} - \frac{\sin(2-\xi)\pi}{2-\xi})$$

$$= \frac{2\sin\pi\xi}{4-\xi^2}.$$

Thus

$$c_X(\xi) = \frac{1}{\pi}(\frac{\sin\xi}{\xi} - \frac{2\sin\pi\xi}{4-\xi^2}).$$

4. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. A simple financial derivative of European type with the payoff function $g \in \mathcal{P}$ has the price

$$\Pi_{g(S(T))}(t) = e^{-r\tau}E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})\right]$$

at time t, where $s = S(t)$ is the stock price at time t and $G \in N(0, 1)$.

(a) A European call has the strike price K and determination date T. Show that the call price at time t equals $s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau)$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$.

(b) Show that the delta of the call in Part (a) equals $\Phi(d_1)$.

5. (Black-Scholes model) Suppose the value of one US dollar at time t equals $\xi(t)$ Swedish crowns and that the price process $(\xi(t))_{0\leq t\leq T}$ is a geometric Brownian motion with volatility σ. Moreover, denote by r_f and r the US and Swedish interest rates, respectively.

Consider the right but not the obligation to buy one US dollar at the price K Swedish crowns at time T. Derive the price in Swedish crowns of this derivative at time t.