SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])

September 1, 2012, morning, v.
No aids.
Each problem is worth 3 points.
Questions on the exam: Christer Borell 0705 292322

1. (Binomial model with $T = 2$, $u = -d > 0$, and $e^r = \frac{1}{2}(e^u + e^d)$) A derivative of European type pays the amount

$$Y = \mid \frac{S(T)}{S(0)} - 1 \mid$$

at time of maturity T. Find $\Pi_Y(0)$.

Solution. We have that

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{\frac{1}{2}(e^u + e^d) - e^d}{e^u - e^d} = \frac{1}{2}$$

and if $v(t) = \Pi_Y(t)$,

$$\begin{cases} v(2)_{|X_1=u,\ X_2=u} = e^{2u} - 1 \\ v(2)_{|X_1=u,\ X_2=d} = 0 \\ v(2)_{|X_1=d,\ X_2=u} = 0 \\ v(2)_{|X_1=d,\ X_2=d} = 1 - e^{-2u}. \end{cases}$$

Now

$$\begin{cases} v(1)_{|X_1=u} = \frac{e^{-r}}{2}(e^{2u} - 1) \\ v(1)_{|X_1=d} = \frac{e^{-r}}{2}(1 - e^{-2u}) \end{cases}$$

and

$$\Pi_Y(0) = e^{-r}(\frac{e^{-r}}{4}(e^{2u} - 1) + \frac{e^{-r}}{4}(1 - e^{-2u}))$$

$$= \frac{e^{-2r}}{4}(e^{2u} - e^{-2u}) = \frac{e^{2u} - e^{-2u}}{(e^u + e^{-u})^2} = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

2. Let $Z(t) = (Z_1(t), Z_2(t))$, $t \geq 0$, be a standard Brownian motion in the plane. Find
$$\mathrm{Var}(e^{Z_1(t)} - e^{Z_2(t)}).$$

Solution. The random variables $Z_1(t), Z_2(t) \in N(0,t)$ are independent and
$$E\left[e^{aG}\right] = e^{\frac{a^2}{2}}$$
if $G \in N(0,1)$ and $a \in \mathbf{R}$. Accordingly from these properties,
$$E\left[e^{Z_1(t)} - e^{Z_2(t)}\right] = e^{\frac{t}{2}} - e^{\frac{t}{2}} = 0$$
and
$$E\left[(e^{Z_1(t)} - e^{Z_2(t)})^2\right] = E\left[e^{2Z_1(t)}\right] - 2E\left[e^{Z_1(t)}e^{Z_2(t)}\right] + E\left[e^{2Z_2(t)}\right]$$
$$= 2e^{2t} - 2E\left[e^{Z_1(t)}\right]E\left[e^{Z_2(t)}\right] = 2e^{2t} - 2e^{t}.$$
The above formulas give
$$\mathrm{Var}(e^{Z_1(t)} - e^{Z_2(t)}) = 2e^{t}(e^{t} - 1).$$

Alternative solution. Since $e^{Z_1(t)}$ and $-e^{Z_2(t)}$ are independent
$$\mathrm{Var}(e^{Z_1(t)} - e^{Z_2(t)}) = \mathrm{Var}(e^{Z_1(t)}) + \mathrm{Var}(-e^{Z_2(t)})$$
$$= 2\mathrm{Var}(e^{Z_1(t)}) = 2(E\left[e^{2Z_1(t)}\right] - (E\left[e^{Z_1(t)}\right])^2)$$
$$= 2e^{2t} - 2e^{t} = 2e^{t}(e^{t} - 1).$$

3. (Black-Scholes model) A stock price process $(S(t))_{t\geq 0}$ is governed by the equation
$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)},\ t \geq 0,$$
where $\mu > r$. If T and K denote strictly positive real numbers, show that
$$E\left[(S(T) - K)^{+}\right] > e^{rT}c(0, S(0), K, T).$$

Solution. If $a > 0$ and

$$f(x,a) = (S(0)e^{(a-\frac{\sigma^2}{2})T-\sigma\sqrt{T}x} - K)^+, \ x \in \mathbf{R},$$

then

$$E\left[(S(T)-K)^+\right] = \int_{-\infty}^{\infty} f(x,\mu)\varphi(x)dx$$

and

$$e^{rT}c(0,S(0),K,T) = \int_{-\infty}^{\infty} f(x,r)\varphi(x)dx.$$

Since $\mu > r$ we have $f(x,\mu) \geq f(x,r)$ with strict inequality if

$$x < \frac{1}{\sigma\sqrt{T}}(\ln\frac{S(0)}{K} + (\mu - \frac{\sigma^2}{2})T).$$

Hence

$$\int_{-\infty}^{\infty} f(x,\mu)\varphi(x)dx > \int_{-\infty}^{\infty} f(x,r)\varphi(x)dx$$

which proves that

$$E\left[(S(T)-K)^+\right] > e^{rT}c(0,S(0),K,T).$$

Alternative solution. For any strictly positive real number a the Black-Scholes theory yields

$$f(a) =_{def} e^{-aT}\int_{-\infty}^{\infty}(S(0)e^{(a-\frac{\sigma^2}{2})T-\sigma\sqrt{T}x} - K)^+dx$$

$$= S(0)\Phi(d_1(a)) - Ke^{-aT}\Phi(d_2(a))$$

where

$$d_1(a) = \frac{\ln\frac{S(0)}{K} + (a+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and

$$d_2(a) = \frac{\ln\frac{S(0)}{K} + (a-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1(a) - \sigma\sqrt{T}.$$

Hence

$$f'(a) = S(0)\varphi(d_1(a))\frac{1}{\sigma\sqrt{T}} - Ke^{-aT}\varphi(d_2(a))\frac{1}{\sigma\sqrt{T}} + KTe^{-aT}\Phi(d_2(a)).$$

But

$$S(0)\varphi(d_1(a)) - Ke^{-aT}\varphi(d_2(a)) = 0$$

(cf the proof of Theorem 5.3.1 in the textbook) and we get

$$f'(a) = KTe^{-aT}\Phi(d_2(a)).$$

Hence

$$\frac{d}{da}(e^{aT}f(a)) = Te^{aT}f(a) + KT\Phi(d_2(a)) > 0$$

and if $\mu > r$, we get

$$E\left[(S(T)-K)^+\right] = e^{\mu T}f(\mu) > e^{rT}f(r) = e^{rT}c(0, S(0), K, T).$$

4. (Dominance principle) Show that the map

$$K \to c(t, S(t), K, T),\ \ K > 0$$

is convex.

5. (Black-Scholes model) Suppose K, T, and σ are strictly positive real numbers.

(a) Let $S = (S(t))_{t\geq 0}$ be a stock price process with volatility σ. State the price of a European call on S with maturity T and strike price K.

(b) Suppose the value of one US dollar at time t equals $\xi(t)$ Swedish crowns and that the price process $(\xi(t))_{0\leq t\leq T}$ is a geometric Brownian motion with volatility σ. Moreover, denote by r_f and r the US and Swedish interest rates, respectively.

Consider the right but not the obligation to buy one US dollar at the price K Swedish crowns at time T. Use Part (a) to derive the price of this derivative at time t expressed in Swedish crowns.