SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])
May 27, 2013, morning, v
No aids.
Questions on the exam: Christer Borell, telephone number 0705 292322
Each problem is worth 3 points.

1. (Binomial Model: $S(0) = B(0) = 1$, $T = 2$, $u = -d = \ln 2$, and $r = 0$). A European-style financial derivative pays the amount Y at time of maturity $T = 2$, where

$$Y = \begin{cases} 0 \text{ if } S(0) = S(2), \\ S(1) \text{ if } S(0) \neq S(2). \end{cases}$$

(a) State the time zero price $\Pi_Y(0)$ of the derivative.
(b) The portfolio strategy h replicates Y. State $h(0) = (h_S(0), h_B(0))$.
(Please, do not hand in any solutions, just answers!)

Solution. (a) We have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1}{3}$$

and

$$q_d = \frac{e^u - e^r}{e^u - e^d} = \frac{2}{3}.$$

Moreover, if $v(t) = \Pi_Y(t)$ and $s = S(0)$,

$$\begin{cases} v(2)_{|X_1=u,X_2=u} = se^u \\ v(2)_{|X_1=u,X_2=d} = 0 \\ v(2)_{|X_1=d,X_2=u} = 0 \\ v(2)_{|X_1=d,X_2=d} = se^d \end{cases}$$

and, hence,

$$\begin{cases} v(1)_{|X_1=u} = e^{-r}q_u se^u = q_u se^{u-r} \\ v(1)_{|X_1=d} = e^{-r}q_d se^d = q_d se^{d-r}. \end{cases}$$

Now

$$\Pi_Y(0) = e^{-r}(q_u^2 se^{u-r} + q_d^2 se^{d-r}) = se^{-2r}(q_u^2 e^u + q_d^2 e^d) = \frac{4}{9}s = \frac{4}{9}.$$

(c) We have

$$\begin{cases} h_S(0)se^u + h_B(0)B(0)e^r = q_u se^{u-r} \\ h_S(0)se^d + h_B(0)B(0)e^r = q_d se^{d-r}. \end{cases}$$

Thus

$$h_S(0) = e^{-r}\frac{q_u e^u - q_d e^d}{e^u - e^d} = \frac{2}{9}$$

and

$$h_B(0) = s\frac{e^{u+d-2r}}{B(0)}\frac{q_d - q_u}{e^u - e^d} = \frac{2}{9}s = \frac{2}{9}.$$

2. (Black-Scholes Model) A European-style financial derivative has at time zero the price a and pays the amount

$$Y = \begin{cases} a + \xi \text{ if } S(T) \geq S(0) \\ a \text{ if } S(T) < S(0). \end{cases}$$

at time of maturity T, where a and T are given positive numbers and ξ is an unknown real number. Find ξ.

Solution. Put $Z = H(S(T) - S(0))$, where H is the Heaviside function. Now $Y = a + \xi Z$ and

$$a = ae^{-rT} + \xi \Pi_Z(0).$$

Thus

$$\xi = \frac{a(1 - e^{-rT})}{\Pi_Z(0)}.$$

Moreover, if $s = S(0)$,

$$\Pi_Z(0) = e^{-rT}E\left[H(s(e^{(r-\frac{\sigma^2}{2})T - \sigma\sqrt{T}G} - 1))\right],$$

where $G \in N(0,1)$ and, hence,

$$\Pi_Z(0) = e^{-rT}P\left[G \leq (\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{T}\right] = e^{-rT}\Phi((\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{T}).$$

Summing up, we have

$$\xi = \frac{a(e^{rT} - 1)}{\Phi((\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{T})}.$$

3. (Black-Scholes Model) Suppose $K, T > 0$ and $N \in \mathbf{N}_+$ are given and consider a European-style derivative which pays the amount

$$Y = \left(\left(\prod_{j=1}^{N} S(\frac{jT}{N}) \right)^{\frac{1}{N}} - K \right)^+$$

at time of maturity T. Find the time zero price $\Pi_Y(0)$ of the derivative. (Hint: $1^2 + 2^2 + ... + N^2 = \frac{1}{6}N(N+1)(2N+1)$)

Solution. Set $S(0) = s$ to get

$$\Pi_Y(0) = e^{-rT} E \left[\left(s \left(\prod_{j=1}^{N} e^{(r-\frac{\sigma^2}{2})\frac{jT}{N} + \sigma W(\frac{jT}{N})} \right)^{\frac{1}{N}} - K \right)^+ \right]$$

$$= e^{-rT} E \left[\left(s e^{(r-\frac{\sigma^2}{2})\frac{(N+1)T}{2N} + \frac{\sigma}{N} \sum_{j=1}^{N} W(\frac{jT}{N})} - K \right)^+ \right].$$

Set $X = \sum_{k=1}^{N} W(\frac{kT}{N})$. Clearly, X is a centred Gaussian random variable and to find its variance put

$$Z_j = W(\frac{jT}{N}) - W(\frac{(j-1)T}{N}),\ j = 1, ..., N.$$

Then

$$X = \sum_{j=1}^{N} (\sum_{i=1}^{k} Z_i) = \sum_{\substack{1 \le i \le j \\ 1 \le j \le N}} Z_i = \sum_{i=1}^{N} (N - i + 1) Z_i$$

and

$$\text{Var}(X) = \sum_{i=1}^{N} (N - i + 1)^2 \text{Var}(Z_i)$$

$$= \frac{T}{N} \sum_{i=1}^{N} (N - i + 1)^2 = \frac{T}{6}(N+1)(2N+1).$$

Thus

$$\Pi_Y(0) = e^{-rT} E\left[\left(se^{(r-\frac{\sigma^2}{2})\frac{(N+1)T}{2N}+\frac{\sigma}{N}\sqrt{\frac{(N+1)(2N+1)T}{6}}G} - K\right)^+\right],$$

where $G \in N(0,1)$. Now put

$$\begin{cases} a = (r-\frac{\sigma^2}{2})\frac{(N+1)T}{2N} \\ b = \frac{\sigma}{N}\sqrt{\frac{(N+1)(2N+1)T}{6}} \end{cases}$$

so that

$$\Pi_Y(0) = e^{-rT} E\left[\left(se^{a-bG} - K\right)^+\right]$$

$$= e^{-rT}\left(se^a \int_{-\infty}^{c} e^{-bx-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}} - K\int_{-\infty}^{c} e^{-bx-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}}\right)$$

where

$$c = \frac{\ln\frac{s}{K} + a}{b}.$$

Summing up, we get

$$\Pi_Y(0) = e^{-rT}\left(se^{a+\frac{b^2}{2}}\Phi(c+b) - K\Phi(c)\right)$$

with a, b, and c defined as above.

4. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.

5. (Dominance Principle) Show that the map

$$K \to c(t, S(t), K, T),\ K > 0$$

is convex.