SOLUTIONS
OPTIONS AND MATHEMATICS
(CTH[$mve095$], GU[$MMA700$])
August 31, 2013, morning, v
No aids.
Questions on the exam: Christer Borell, telephone number 0705 292322
Each problem is worth 3 points.

1. The random variables X and Y are independent and uniformly distributed on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Find the density function and characteristic function of the random variable $X+Y$.

Solution. The density function of X and Y equals

$$f(x) = \begin{cases} 1 \text{ if } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 0 \text{ otherwise,} \end{cases}$$

and we conclude that the density function g of $X+Y$ is even and

$$g(x) = \int_{-\infty}^{\infty} f(x-y) f(y) dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-y) dy$$

$$= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(t) dt = \begin{cases} 0 \text{ if } x > 1, \\ 1-x \text{ if } 0 \leq x \leq 1. \end{cases}$$

Thus $g(x) = (1- \mid x \mid)^+$ for every real number x.

The independence of X and Y implies that

$$c_{X+Y}(\xi) = E\left[e^{i\xi(X+Y)}\right] = E\left[e^{i\xi X} e^{i\xi Y}\right] = E\left[e^{i\xi X}\right] E\left[e^{i\xi Y}\right]$$

$$= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\xi x} dx\right)^2 = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \xi x dx\right)^2 = \frac{4\sin^2(\xi/2)}{\xi^2} \text{ if } \xi \in \mathbf{R} \setminus \{0\}$$

and, in addition, $c_{X+Y}(0) = 1$.

2. (Binomial Model: T periods and $d < r < u$) A European-style financial derivative pays the amount

$$Y = \sum_{t=1}^{T} (S(t) - S(t-1))^+$$

at time of maturity T. Find the time zero price $\Pi_Y(0)$ of the derivative.

Solution. As usual let

$$q_u = \frac{e^r - e^d}{e^u - e^d} \text{ and } q_d = \frac{e^u - e^r}{e^u - e^d}.$$

Put $Y_i = (S(i) - S(i-1))^+$, $i = 1, ..., T$. A European-style derivative paying Y_i at time T has the price $\Pi_{Y_i}(i) = Y_i e^{-r(T-i)}$ at time i and the price

$$\Pi_{Y_i}(i-1)$$

$$= e^{-r(T+1-i)} \left\{ q_u(S(i-1)e^u - S(i-1))^+) + q_d((S(i-1)e^d - S(i-1))^+) \right\}$$

at time $i - 1$. Note that

$$\Pi_{Y_i}(i-1) = Ae^{-r(T+1-i)} S(i-1)$$

where

$$A = q_u(e^u - 1)^+ + q_d(e^d - 1)^+.$$

Hence

$$\Pi_{Y_i}(0) = Ae^{-r(T+1-i)} S(0)$$

and

$$\Pi_Y(0) = AS(0) \sum_{i=1}^{T} e^{-r(T+1-i)} = A \frac{1 - e^{-rT}}{e^r - 1} S(0).$$

3. (Black-Scholes Model) Suppose a, T, and K are given positive numbers. A European-style financial derivative has the payoff

$$Y = \min((S(T) - K)^+, (K + 2a - S(T))^+)$$

at time of maturity T. Moreover suppose $0 \leq t < T$ and $\tau = T - t$. (a) Find $\Delta(t)$ (= the delta of the derivative at time t). (b) Show that $\Delta(t) \geq 0$ if $K \geq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$ and $\Delta(t) \leq 0$ if $K + 2a \leq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$.

Solution. First note that

$$Y = \min(a, (S(T) - K)^+ - \min(a, (S(T) - K - a)^+)$$

$$= (S(T) - K)^+ - 2(S(T) - K - a)^+ + (S(T) - K - 2a)^+.$$

We next introduce

$$d_1(x) = \frac{1}{\sigma\sqrt{\tau}}(\ln \frac{S(t)}{x} + (r + \frac{\sigma^2}{2})\tau), \; x > 0,$$

and use the known delta of a European-style call to get

$$\Delta(t) = \Phi(d_1(K)) - 2\Phi(d_1(K + a)) + \Phi(d_1(K + 2a)).$$

Now introduce the function $f(x) = \Phi(d_1(x))$, $x > 0$, and note that

$$\Delta(t) = f(K) - 2f(K + a) + f(K + 2a).$$

Moreover,

$$f'(x) = -\varphi(d_1(x))\frac{1}{\sigma\sqrt{\tau}x}$$

and

$$f''(x) = -d_1(x)\varphi(d_1(x))\frac{1}{(\sigma\sqrt{\tau}x)^2} + \varphi(d_1(x))\frac{1}{\sigma\sqrt{\tau}x^2}$$

$$= \varphi(d_1(x))\frac{1}{(\sigma\sqrt{\tau}x)^2}(-d_1(x) + \sigma\sqrt{\tau}).$$

Thus $f''(x) \geq 0$ if $d_1(x) \leq \sigma\sqrt{\tau}$ or, stated more explicitely, $f''(x) \geq 0$ if $x \geq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$. Therefore $f(x)$ is convex for $x \geq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$ and we conclude that $\Delta(t) \geq 0$ if $K \geq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$. In a similar way $f''(x) \leq 0$ if $x \leq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$ and it follows that $\Delta(t) \leq 0$ if $K + 2a \leq S(t)e^{(r-\frac{\sigma^2}{2})\tau}$.

4. (Dominance Principle) State and prove the Put-Call Parity relation.

5. (Black-Scholes Model) Let

$$S(t) = S(0)e^{\alpha t + \sigma W(t)},\ t \geq 0,$$

and suppose $0 < t_1 < ... < t_n$ and $a_1 < b_1, ..., a_n < b_n$. Prove that

$$P\left[a_1 < S(t_1) < b_1, ..., a_n < S(t_n) < b_n\right]$$

$$= \int \cdots \int_{A_1 \times ... \times A_n} \prod_{k=1}^{n} \left\{ \frac{1}{\sqrt{2\pi(t_k - t_{k-1})}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}} \right\} dx_1 ... dx_n,$$

where $x_0 = 0,\ t_0 = 0$, and

$$A_k = \left] \frac{1}{\sigma}(\ln \frac{a_k}{S(0)} - \alpha t_k), \frac{1}{\sigma}(\ln \frac{b_k}{S(0)} - \alpha t_k) \right[,\ k = 1, ..., n.$$