Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]). May 26^{th}, 2014

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REMARK: No aids permitted

1. Consider an American derivative with intrinsic value

$$Y(t) = \min(S(t), (24 - S(t))_+)$$

and expiring at time $T = 3$. The initial price of the underlying stock is $S(0) = 27$, while at future times it follows the binomial model

$$S(t+1) = \begin{cases} \frac{4}{3}S(t) & \text{with probability } 1/2 \\ \frac{2}{3}S(t) & \text{with probability } 1/2 \end{cases}$$

for $t = 0, 1, 2$. Assume also that the interest rate of the bond is zero. Compute the possible paths of the fair value of the derivative (max. 2 points). In which case it is optimal for the buyer to exercise the derivative prior to expiration? (max. 1 point). Compute the expected return of a constant portfolio which consists of a long position in one share of the stock and a short position in one share of the derivative (max. 2 points).

Solution: With the given values of the parameters u, d, r, we have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{1}{2} = q_d.$$

The fair price $\hat{\Pi}_Y(t)$ of the American derivative satisfies

$$\begin{aligned}\hat{\Pi}_Y(t) &= \max(Y(t), e^{-r}(q_u \hat{\Pi}_Y^u(t+1) + q_d \hat{\Pi}_Y^d(t+1))) \\ &= \max(Y(t), \frac{1}{2}(\hat{\Pi}_Y^u(t+1) + \hat{\Pi}_Y^d(t+1))),\end{aligned}$$

where $\hat{\Pi}_Y^u(t)$ (resp. $\hat{\Pi}_Y^d(t)$) is the price of the derivative at time t assuming that the stock price goes up (resp. down) at time t. The diagram of the stock price is

$$
\begin{tikzcd}
 & & & S(3)=64 \\
 & & S(2)=48 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=36 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=32 \\
S(0)=27 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=24 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=18 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=16 \\
 & & S(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=8
\end{tikzcd}
$$

to which there corresponds the following diagram for the intrinsic value:

$$
\begin{tikzcd}
 & & & Y(3)=0=\hat{\Pi}_Y(3) \\
 & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=0=\hat{\Pi}_Y(3) \\
Y(0)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=6 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=8=\hat{\Pi}_Y(3) \\
 & & Y(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & Y(3)=8=\hat{\Pi}_Y(3)
\end{tikzcd}
$$

Therefore

$$S(2)=48 \Rightarrow \hat{\Pi}_Y(2)=0, \quad S(2)=24 \Rightarrow \hat{\Pi}_Y(2)=4, \quad S(2)=12 \Rightarrow \hat{\Pi}_Y(2)=12$$

$$S(1) = 36 \Rightarrow \hat{\Pi}_Y(1) = 2, \quad S(1) = 18 \Rightarrow \hat{\Pi}_Y(1) = 8,$$

and $\hat{\Pi}_Y(0) = 5$. We thereby obtained the following diagram for the price of the derivative:

$$\begin{tikzcd}
 & & & \hat{\Pi}_Y(3) = 0 \\
 & & \hat{\Pi}_Y(2) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \hat{\Pi}_Y(1) = 2 \arrow[ur, "u"] \arrow[dr, "d"] & & \hat{\Pi}_Y(3) = 0 \\
\hat{\Pi}_Y(0) = 5 \arrow[ur, "u"] \arrow[dr, "d"] & & \hat{\Pi}_Y(2) = 4 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \hat{\Pi}_Y(1) = 8 \arrow[ur, "u"] \arrow[dr, "d"] & & \hat{\Pi}_Y(3) = 8 \\
 & & \boxed{\hat{\Pi}_Y(2) = 12} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \hat{\Pi}_Y(3) = 8
\end{tikzcd}$$

This completes the first part of the exercise (2 points). The only case in which the price of the derivative equals its intrinsic value prior to expiration is at time $t = 2$ when the price of the stock is $S(2) = 12$ (i.e., the stock price goes down in the first two steps). This is indicated in the previous diagram by putting the price of the derivative in a box. In this case, and only in this case, it is optimal to exercise the derivative prior to expiration. This completes the second part of the exercise (1 point). Let $V(t)$ be the value of a portfolio with 1 share of the stock and -1 share of the derivative. The return of the portfolio is the random variable

$$R = \frac{V(3) - V(0)}{V(0)} = \frac{V(3)}{V(0)} - 1.$$

The expected return is

$$\mathbb{E}[R] = \frac{1}{V(0)}\mathbb{E}[V(3)] - 1.$$

We have

$$V(0) = S(0) - \hat{\Pi}_Y(0) = 22$$

and

$$V(3) = \begin{cases} 64 & \text{with probability } 1/8 \\ 32 & \text{with probability } 3/8 \\ 8 & \text{with probability } 3/8 \\ 0 & \text{with probability } 1/8 \end{cases}$$

Hence $\mathbb{E}[V(3)] = 64 \cdot \frac{1}{8} + 32 \cdot \frac{3}{8} + 8 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} = 23$. Therefore

$$\mathbb{E}[R] = \frac{23}{22} - 1 = \frac{1}{22} \approx 4.54\%,$$

which completes the third part of the exercise (2 points).

2. Consider a European derivative with pay-off $Y = S(T)(S(T)-K)$ and time of maturity T, where $K > 0$ is a constant. It is assumed that the price $S(t)$ of the underlying stock follows a geometric Brownian motion for $t \in [0,T]$ and that the interest rate of the bond is a constant $r > 0$. Compute the Black-Scholes price $\Pi_Y(t)$ of this derivative (max. 2 points) and the hedging portfolio $h(t) = (h_S(t), h_B(t))$ (max. 1 point). Finally, assume $S(0) = K$ and compute the expected return of a constant portfolio with 1 share of this derivative (max. 2 points)

   Solution: The pay-off function is $g(z) = z(z-K)$; the Black-Scholes price is given by $\Pi_Y(t) = v(t, S(t))$, where

$$\begin{aligned} v(t,s) &= e^{-r\tau} \int_{\mathbb{R}} g(se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{\mathbb{R}} \left[se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} \right] \left[se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} - K \right] e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= s \left[se^{(r+\sigma^2)\tau} - K \right]. \end{aligned}$$

   This solves the first part of the exercise (2 points). The number of shares of the underlying stock in the hedging portfolio is given by $h_S(t) = \Delta(t, S(t))$, where

$$\Delta(t,s) = \frac{\partial v}{\partial s} = 2se^{(r+\sigma^2)\tau} - K.$$

   The number of shares of the bond is obtained using

$$\Pi_Y(t) = h_S(t)S(t) + h_B(t)B(t) \Rightarrow h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) = -\frac{e^{-rt}}{B(0)} e^{(r+\sigma^2)\tau} S(t)^2.$$

   This completes the second part of the exercise (1 point). The return of the portfolio is

$$R = \frac{\Pi_Y(T)}{\Pi_Y(0)} - 1.$$

Using $\Pi_Y(T) = S(T)(S(T) - K)$ and $\Pi_Y(0) = K^2(e^{(r+\sigma^2)T} - 1)$, we obtain

$$\mathbb{E}[R] = \frac{\mathbb{E}[S(T)(S(T) - K)]}{K^2(e^{(r+\sigma^2)T} - 1)} - 1.$$

Writing the geometric Brownian motion at time T as

$$S(T) = S(0)e^{(\mu - \frac{\sigma^2}{2})T + \sigma W(T)} = Ke^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}G},$$

where $G = W(T)/\sqrt{T} \in N(0, 1)$, we get

$$\mathbb{E}[S(T)^2] = K^2 e^{(2\mu - \sigma^2)T} \int_{\mathbb{R}} e^{2\sigma\sqrt{T}x - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = K^2 e^{(2\mu + \sigma^2)T},$$

$$\mathbb{E}[S(T)] = Ke^{(\mu - \frac{\sigma^2}{2})T} \int_{\mathbb{R}} e^{\sigma\sqrt{T}x - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = Ke^{\mu T}.$$

Therefore

$$\mathbb{E}[S(T)(S(T) - K)] = \mathbb{E}[S(T)^2] - K\mathbb{E}[S(T)] = K^2 e^{\mu T}(e^{(\mu+\sigma^2)T} - 1).$$

We conclude that the expected return is given by

$$\mathbb{E}[R] = \frac{e^{\mu T}(e^{(\mu+\sigma^2)T} - 1)}{e^{(r+\sigma^2)T} - 1} - 1.$$

This completes the third part of the exercise (2 points).

3. Assume that the dominance principle holds. Let $p(t, S(t), K, T)$ be the value at time t of a European put option with strike price K and maturity time T on a stock with price $S(t)$. Let $c(t, S(t), K, T)$ be the price of the corresponding European call option, while $C(t, S(t), K, T)$ and $P(t, S(t), K, T)$ denote the price of the American call and put with the same parameters. Give a complete proof of the following facts: (i) The put-call parity (max. 1 point); (ii) The price of the European call is a non-decreasing function of the time of maturity (max. 2 points); (iii) $C(t, S(t), K, T) > S(t) - K$, for $t \in [0, T)$ (max. 1 point). Explain why (iii) implies that the American and the European call have the same value (max. 1 point).

   Solution: See Borell's book *Introduction to the Black-Scholes theory*, Theorems 1.1.2, 1.1.3.