

Exam for the course “Options and Mathematics”  
(CTH[*MVE095*], GU[*MMA700*]). May 26<sup>th</sup>, 2014

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REMARK: No aids permitted

1. Consider an American derivative with intrinsic value

$$Y(t) = \min(S(t), (24 - S(t))_+)$$

and expiring at time  $T = 3$ . The initial price of the underlying stock is  $S(0) = 27$ , while at future times it follows the binomial model

$$S(t+1) = \begin{cases} \frac{4}{3}S(t) & \text{with probability } 1/2 \\ \frac{2}{3}S(t) & \text{with probability } 1/2 \end{cases}$$

for  $t = 0, 1, 2$ . Assume also that the interest rate of the bond is zero. Compute the possible paths of the fair value of the derivative (max. 2 points). In which case it is optimal for the buyer to exercise the derivative prior to expiration? (max. 1 point). Compute the expected return of a constant portfolio which consists of a long position in one share of the stock and a short position in one share of the derivative (max. 2 points).

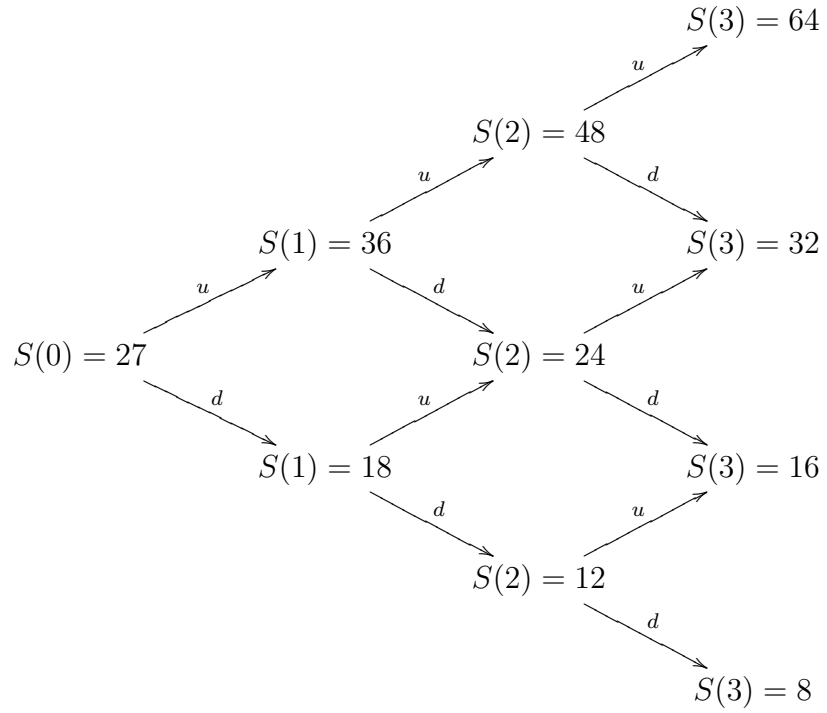
**Solution:** With the given values of the parameters  $u, d, r$ , we have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{1}{2} = q_d.$$

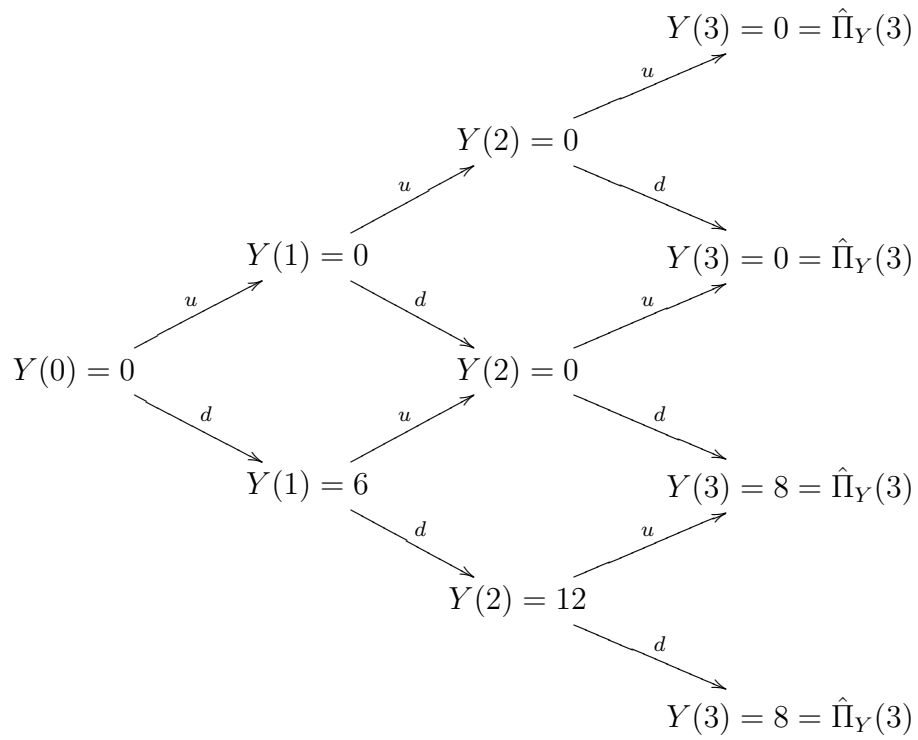
The fair price  $\hat{\Pi}_Y(t)$  of the American derivative satisfies

$$\begin{aligned} \hat{\Pi}_Y(t) &= \max(Y(t), e^{-r}(q_u \hat{\Pi}_Y^u(t+1) + q_d \hat{\Pi}_Y^d(t+1))) \\ &= \max(Y(t), \frac{1}{2}(\hat{\Pi}_Y^u(t+1) + \hat{\Pi}_Y^d(t+1))), \end{aligned}$$

where  $\hat{\Pi}_Y^u(t)$  (resp.  $\hat{\Pi}_Y^d(t)$ ) is the price of the derivative at time  $t$  assuming that the stock price goes up (resp. down) at time  $t$ . The diagram of the stock price is



to which there corresponds the following diagram for the intrinsic value:

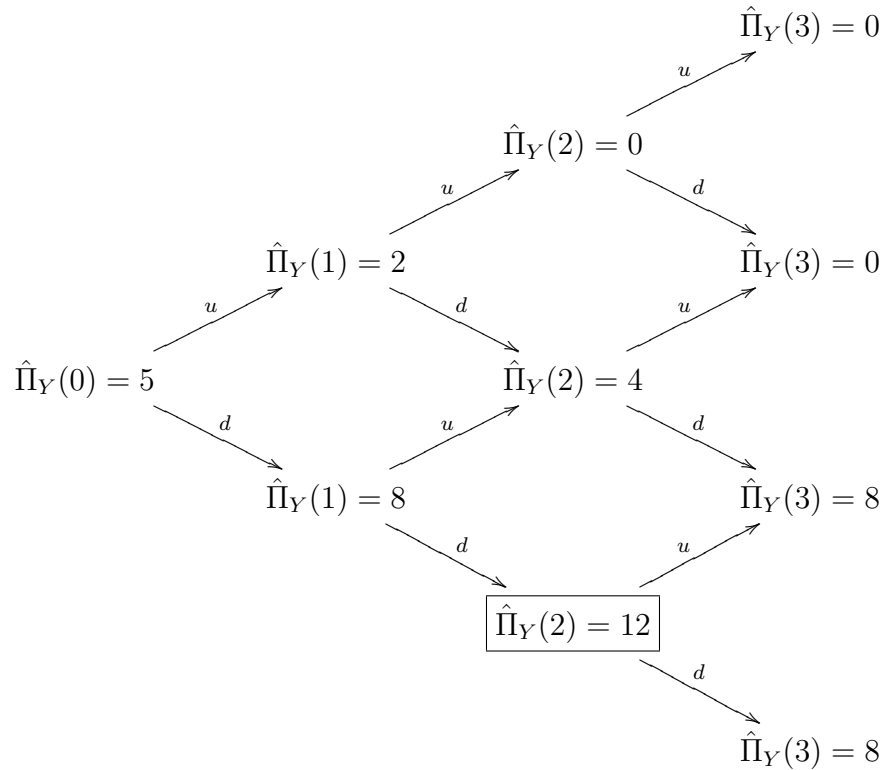


Therefore

$$S(2) = 48 \Rightarrow \hat{\Pi}_Y(2) = 0, \quad S(2) = 24 \Rightarrow \hat{\Pi}_Y(2) = 4, \quad S(2) = 12 \Rightarrow \hat{\Pi}_Y(2) = 12$$

$$S(1) = 36 \Rightarrow \hat{\Pi}_Y(1) = 2, \quad S(1) = 18 \Rightarrow \hat{\Pi}_Y(1) = 8,$$

and  $\hat{\Pi}_Y(0) = 5$ . We thereby obtained the following diagram for the price of the derivative:



This completes the first part of the exercise (2 points). The only case in which the price of the derivative equals its intrinsic value prior to expiration is at time  $t = 2$  when the price of the stock is  $S(2) = 12$  (i.e., the stock price goes down in the first two steps). This is indicated in the previous diagram by putting the price of the derivative in a box. In this case, and only in this case, it is optimal to exercise the derivative prior to expiration. This completes the second part of the exercise (1 point). Let  $V(t)$  be the value of a portfolio with 1 share of the stock and -1 share of the derivative. The return of the portfolio is the random variable

$$R = \frac{V(3) - V(0)}{V(0)} = \frac{V(3)}{V(0)} - 1.$$

The expected return is

$$\mathbb{E}[R] = \frac{1}{V(0)} \mathbb{E}[V(3)] - 1.$$

We have

$$V(0) = S(0) - \hat{\Pi}_Y(0) = 22$$

and

$$V(3) = \begin{cases} 64 & \text{with probability } 1/8 \\ 32 & \text{with probability } 3/8 \\ 8 & \text{with probability } 3/8 \\ 0 & \text{with probability } 1/8 \end{cases}$$

Hence  $\mathbb{E}[V(3)] = 64 \cdot \frac{1}{8} + 32 \cdot \frac{3}{8} + 8 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} = 23$ . Therefore

$$\mathbb{E}[R] = \frac{23}{22} - 1 = \frac{1}{22} \approx 4.54\%,$$

which completes the third part of the exercise (2 points).

2. Consider a European derivative with pay-off  $Y = S(T)(S(T) - K)$  and time of maturity  $T$ , where  $K > 0$  is a constant. It is assumed that the price  $S(t)$  of the underlying stock follows a geometric Brownian motion for  $t \in [0, T]$  and that the interest rate of the bond is a constant  $r > 0$ . Compute the Black-Scholes price  $\Pi_Y(t)$  of this derivative (max. 2 points) and the hedging portfolio  $h(t) = (h_S(t), h_B(t))$  (max. 1 point). Finally, assume  $S(0) = K$  and compute the expected return of a constant portfolio with 1 share of this derivative (max. 2 points)

**Solution:** The pay-off function is  $g(z) = z(z - K)$ ; the Black-Scholes price is given by  $\Pi_Y(t) = v(t, S(t))$ , where

$$\begin{aligned} v(t, s) &= e^{-r\tau} \int_{\mathbb{R}} g(se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{\mathbb{R}} [se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x}] [se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} - K] e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= s [se^{(r+\sigma^2)\tau} - K]. \end{aligned}$$

This solves the first part of the exercise (2 points). The number of shares of the underlying stock in the hedging portfolio is given by  $h_S(t) = \Delta(t, S(t))$ , where

$$\Delta(t, s) = \frac{\partial v}{\partial s} = 2se^{(r+\sigma^2)\tau} - K.$$

The number of shares of the bond is obtained using

$$\Pi_Y(t) = h_S(t)S(t) + h_B(t)B(t) \Rightarrow h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) = -\frac{e^{-rt}}{B(0)}e^{(r+\sigma^2)\tau}S(t)^2.$$

This completes the second part of the exercise (1 point). The return of the portfolio is

$$R = \frac{\Pi_Y(T)}{\Pi_Y(0)} - 1.$$

Using  $\Pi_Y(T) = S(T)(S(T) - K)$  and  $\Pi_Y(0) = K^2(e^{(r+\sigma^2)T} - 1)$ , we obtain

$$\mathbb{E}[R] = \frac{\mathbb{E}[S(T)(S(T) - K)]}{K^2(e^{(r+\sigma^2)T} - 1)} - 1.$$

Writing the geometric Brownian motion at time  $T$  as

$$S(T) = S(0)e^{(\mu - \frac{\sigma^2}{2})T + \sigma W(T)} = Ke^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}G},$$

where  $G = W(T)/\sqrt{T} \in N(0, 1)$ , we get

$$\mathbb{E}[S(T)^2] = K^2e^{(2\mu - \sigma^2)T} \int_{\mathbb{R}} e^{2\sigma\sqrt{T}x - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = K^2e^{(2\mu + \sigma^2)T},$$

$$\mathbb{E}[S(T)] = Ke^{(\mu - \frac{\sigma^2}{2})T} \int_{\mathbb{R}} e^{\sigma\sqrt{T}x - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = Ke^{\mu T}.$$

Therefore

$$\mathbb{E}[S(T)(S(T) - K)] = \mathbb{E}[S(T)^2] - K\mathbb{E}[S(T)] = K^2e^{\mu T}(e^{(\mu + \sigma^2)T} - 1).$$

We conclude that the expected return is given by

$$\mathbb{E}[R] = \frac{e^{\mu T}(e^{(\mu + \sigma^2)T} - 1)}{e^{(r + \sigma^2)T} - 1} - 1.$$

This completes the third part of the exercise (2 points).

3. Assume that the dominance principle holds. Let  $p(t, S(t), K, T)$  be the value at time  $t$  of a European put option with strike price  $K$  and maturity time  $T$  on a stock with price  $S(t)$ . Let  $c(t, S(t), K, T)$  be the price of the corresponding European call option, while  $C(t, S(t), K, T)$  and  $P(t, S(t), K, T)$  denote the price of the American call and put with the same parameters. Give a complete proof of the following facts: (i) The put-call parity (max. 1 point); (ii) The price of the European call is a non-decreasing function of the time of maturity (max. 2 points); (iii)  $C(t, S(t), K, T) > S(t) - K$ , for  $t \in [0, T)$  (max. 1 point). Explain why (iii) implies that the American and the European call have the same value (max. 1 point).

**Solution:** See Borell's book *Introduction to the Black-Scholes theory*, Theorems 1.1.2, 1.1.3.