Exam for the course “Options and Mathematics” (CTH[*MVE095*], GU[*MMA700*]). Period 4, 2013/14

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REMARK: No aids permitted

1. Consider a 3-period binomial asset pricing model with the following parameters:

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p_u = \frac{3}{4}.$$

Assuming $S(0) = 27$ and that the bond has zero interest rate ($r = 0$), compute the initial price of the Lookback Option with pay-off

$$Y = S(3) - \min(S(0), S(1), S(2), S(3))$$

and time of maturity $T = 3$ (max. 3 points). Compute the probability that the derivative expires in the money (max. 1 point) and the probability that the return of a constant portfolio with a short position on this derivative be positive (max. 1 point).

Solution: The binomial tree of the stock price is

$$\begin{tikzcd}
 & & & S(3)=64 \\
 & & S(2)=48 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=36 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=32 \\
S(0)=27 \arrow[ur, "u"] \arrow[dr, "d"] & & 24 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=18 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=16 \\
 & & S(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=8
\end{tikzcd}$$

To compute the initial price of a contingent claim it is convenient to use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} q_{x_1} \cdot \dots \cdot q_{x_N} Y(x), \tag{1}$$

where $Y(x)$ denotes the pay-off as a function of the path of the stock price. In this exercise we have $N = 3$, $r = 0$ and

$$q_u = q_d = \frac{1}{2}.$$

So, it remains to compute the pay-off for all possible paths of the stock price, where

$$Y = S(3) - \min(S(0), S(1), S(2), S(3)).$$

For instance

$$Y(u,u,u) = 64 - \min(27, 36, 48, 64) = 64 - 27 = 37.$$

Similarly we find

$$Y(u,u,d) = 5, \qquad Y(u,d,u) = 8, \quad Y(u,d,d) = 0$$

$$Y(d,u,u) = 14, \quad Y(d,u,d) = 0, \quad Y(d,d,u) = 4, \quad Y(d,d,d) = 0.$$

Replacing in (1) we obtain

$$\begin{aligned}\Pi_Y(0) &= (q_u)^3 Y(u,u,u) + (q_u)^2 q_d Y(u,u,d) + (q_u)^2 q_d Y(u,d,u) \\ &\quad + (q_u)^2 q_d Y(d,u,u) + (q_d)^2 q_u Y(d,d,u) \\ &= \left(\frac{1}{2}\right)^3 (37 + 5 + 8 + 14 + 4) = \frac{17}{2} = 8.5.\end{aligned}$$

This concludes the first part of the exercise (3 points). The probability that the derivative expires in the money is the probability that $Y > 0$. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$\begin{aligned}\mathbb{P}(Y > 0) &= \mathbb{P}(\{u,u,u\}) + \mathbb{P}(\{u,u,d\}) + \mathbb{P}(\{u,d,u\}) + \mathbb{P}(\{d,u,u\}) + \mathbb{P}(\{d,d,u\}) \\ &= (p_u)^3 + p_d (p_u)^2 + (p_u)^2 p_d + (p_u)^2 p_d + (p_d)^2 p_u \\ &= (p_u)^3 + 3(p_u)^2 p_d + (p_d)^2 p_u = \frac{27}{64} + 3\frac{9}{16}\frac{1}{4} + \frac{1}{16}\frac{3}{4} = \frac{57}{64} \approx 89\%\end{aligned}$$

This concludes the second part of the exercise (1 point). Next consider a constant portfolio with a short position on the derivative. This means that we buy the derivative at time $t = 0$ and we wait (without changing the portfolio) until the expiration time $t = 3$. The return will be positive if $\Pi_Y(3) < \Pi_Y(0)$. But $\Pi_Y(3) = Y$, which, according to the computations above, is smaller than $\Pi_Y(0) = 8.5$ when the stock price follows one of the paths $\{u,u,d\}$, $\{u,d,u\}$, $\{u,d,d\}$, $\{d,u,d\}$, $\{d,d,u\}$ or $\{d,d,d\}$. Hence

$$\mathbb{P}[\Pi_Y(3) < \Pi_Y(0)] = 2(p_u)^2 p_d + 3(p_d)^2 p_u + (p_d)^3 = \frac{7}{16} \approx 44\%$$

This concludes the third part of the exercise (1 point).

2. Compute the Black-Scholes price of European calls and puts (max. 3 points). Next consider a constant portfolio in the interval $[0, T]$ which is invested in N standard European derivatives with pay-off functions $g_1, \dots g_N$ and time of maturity T. Let the price of these derivatives be given by the Black-Scholes formula. Show that the portfolio satisfies the dominance principle (max. 2 points).
   Solution: See Borell's notes, Theorems 5.2.1 and 5.1.2

3. Consider two stocks with prices

$$S_1(t) = S_1(0)e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_1(t)}, \quad S_2(t) = S_2(0)e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 W_2(t)}$$

   where $\sigma_1, \sigma_2 > 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and $W_1(t), W_2(t)$ are two Brownian motions. Let $T > 0$ and assume that $W_1(T)$ and $W_2(T)$ are independent random variables. Compute the Markowitz portfolio of an investor with initial capital $K > 0$ and risk aversion θ who wants to invest in the stocks and in a money market with interest $r > 0$ during the interval of time $[0, T]$ (max. 5 points)

   Solution: . Let a_1 be the number of shares of the first stock and a_2 the number of shares of the second stock in the Markowitz portfolio and let

$$\pi_1 = \frac{a_1 S_1(0)}{K}, \qquad \pi_2 = \frac{a_2 S_2(0)}{K}.$$

   Then we have the formula

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{2\theta} C^{-1} \begin{pmatrix} \mathbb{E}[R_1] - \rho \\ \mathbb{E}[R_2] - \rho \end{pmatrix} \tag{2}$$

   where R_1, R_2 are the returns of the two stocks, $\rho = e^{rT} - 1$ and C^{-1} is the inverse of the covariant matrix $C_{ij} = C(R_i, R_j)$. The random variables R_1, R_2, are given by

$$R_1 = \frac{S_1(T) - S_1(0)}{S_1(0)}, \quad R_2 = \frac{S_2(T) - S_2(0)}{S_2(0)}.$$

   It follows that

$$\mathbb{E}[R_1] = \frac{1}{S_1(0)} \mathbb{E}[S_1(T)] - 1, \quad \mathrm{Var}[R_1] = \frac{1}{S_1(0)^2} \mathrm{Var}[S_1(T)]$$

   and similarly for the second stock. Since the expectation and the variance of the geometric Brownian motion $S_1(t)$ are given by

$$\mathbb{E}[S_1(t)] = S_1(0)e^{\mu_1 t}, \quad \mathrm{Var}[S_1(t)] = S_1(0)^2 e^{2\mu_1 t}(e^{\sigma_1^2 t} - 1)$$

   we obtain

$$\mathbb{E}[R_1] = e^{\mu_1 T} - 1, \quad \mathrm{Var}[R_1] = e^{2\mu_1 T}(e^{\sigma_1^2 T} - 1).$$

   and similarly for the second stock

$$\mathbb{E}[R_2] = e^{\mu_2 T} - 1, \quad \mathrm{Var}[R_2] = e^{2\mu_2 T}(e^{\sigma_2^2 T} - 1).$$

Since $W_1(T)$ and $W_2(T)$ are independent, then R_1 and R_2 are also independent. Therefore $\mathrm{Cov}(R_1, R_2) = C_{12} = C_{21} = 0$ and so the matrix of covariances of the random variables R_1, R_2 is given by

$$C = \begin{pmatrix} \mathrm{Var}[R_1] & 0 \\ 0 & \mathrm{Var}[R_2] \end{pmatrix} \Rightarrow C^{-1} = \begin{pmatrix} 1/\mathrm{Var}[R_1] & 0 \\ 0 & 1/\mathrm{Var}[R_2] \end{pmatrix}$$

Replacing in (2) we obtain

$$\pi_1 = \frac{1}{2\theta} \frac{\mathbb{E}[R_1] - \rho}{\mathrm{Var}[R_1]} = \frac{1}{2\theta} \frac{e^{\mu_1 T} - e^{rT}}{e^{2\mu_1 T}(e^{\sigma_1^2 T} - 1)}, \quad \pi_2 = \frac{1}{2\theta} \frac{\mathbb{E}[R_2] - \rho}{\mathrm{Var}[R_2]} = \frac{1}{2\theta} \frac{e^{\mu_2 T} - e^{rT}}{e^{2\mu_2 T}(e^{\sigma_2^2 T} - 1)}$$

This completes the exercise (5 points).