Exam for the course "Options and Mathematics" (CTH[$MVE095$], GU[$MMA700$]). Period 4, 2013/14

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REMARK: No aids permitted

1. Assume that the stock price $S(t)$ follows a 1-period binomial model with parameters $u > d$ and that the interest rate of the bond is $r > 0$. Show that there exists no self-financing arbitrage portfolio invested in the stock and the bond in the interval $t \in [0, 1]$ if and only if $d < r < u$ (max 3 points). Show that any derivative on the stock expiring at time $t = 1$ can be hedged in this market (max 2 points).

   Solution: See Theorem 3.2 (step 1) in Ref. [3] and Theorem 3.1 in Ref. [4]

2. Let $c(t)$ denote the Black-Scholes price at time t of a European call with strike $K > 0$ and maturity $T > 0$ on a stock with price $S(t)$ and volatility $\sigma > 0$. Let $r > 0$ denote the interest rate of the bond. Compute the following limits:

$$\lim_{K\to 0^+} c(t), \qquad \lim_{K\to +\infty} c(t), \qquad \lim_{T\to +\infty} c(t), \qquad \lim_{\sigma\to 0^+} c(t), \qquad \lim_{\sigma\to +\infty} c(t).$$

   Each limit gives 1 point if it is correct, 0 otherwise.

   Solution: Recall that

$$c(t, x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{1}$$

   where

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}, \tag{2}$$

   and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$ is the standard normal distribution. As $\sigma \to 0^+$ we have $d_1 \to d_2$ and

$$d_2 \sim \frac{1}{\sqrt{\tau}}(\log\frac{x}{K} + r\tau)\sigma^{-1}.$$

   Hence

$$\begin{aligned} &d_2 \to +\infty, \quad \text{if } x > Ke^{-r\tau}, \\ &d_2 \to -\infty, \quad \text{if } x < Ke^{-r\tau}, \\ &d_2 \to 0, \quad \text{if } x = Ke^{-r\tau}, \end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{\sigma\to 0^+}\Phi(d_1) = \lim_{\sigma\to 0^+}\Phi(d_2) = 1, \quad \text{if } x > Ke^{-r\tau},\\
&\lim_{\sigma\to 0^+}\Phi(d_1) = \lim_{\sigma\to 0^+}\Phi(d_2) = 0, \quad \text{if } x < Ke^{-r\tau},\\
&\lim_{\sigma\to 0^+}\Phi(d_1) = \lim_{\sigma\to 0^+}\Phi(d_2) = \Phi(0), \quad \text{if } x = Ke^{-r\tau}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{\sigma\to 0^+} c(t,x) = x - Ke^{-r\tau} \quad \text{if } x > Ke^{-r\tau},\\
&\lim_{\sigma\to 0^+} c(t,x) = 0, \quad \text{if } x \leq Ke^{-r\tau},
\end{aligned}$$

i.e., $\lim_{\sigma\to 0^+} c(t,x) = (x - Ke^{-r\tau})_+$. For $\sigma \to +\infty$ we have $d_2 \to -\infty$ and $d_1 \to +\infty$, hence $\Phi(d_1) \to 1$ and $\Phi(d_2) \to 0$. Thus $c(t,x) \to x$ as $\sigma \to +\infty$. As $K \to 0^+$, both d_1 and d_2 diverge to $+\infty$, hence

$$\lim_{K\to 0^+} c(t,x) = x.$$

For $K \to +\infty$, d_1, d_2 diverge to $-\infty$. Hence the first term in $c(t,x)$ converges to zero. As the first term in $c(t,x)$ always dominates the second term (since $c(t,x) > 0$), then the second term also goes to zero and thus

$$\lim_{K\to +\infty} c(t,x) = 0.$$

For $T \to +\infty$ we have $d_2 \to -\infty$ and $d_1 \to +\infty$, hence

$$\lim_{T\to +\infty} c(t,x) = x.$$

3. Consider an American put option with strike $K = 3/4$ at the maturity time $T = 2$. Let the price $S(t)$ of the underlying stock be given by the binomial model with parameters

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}.$$

Assume $S(0)$=1. Compute the fair price of the derivative (max 2 points) and the hedging portfolio (max 2 points) at each time $t = 0, 1, 2$. Verify if the put-call parity holds at all times (max 1 point).

Solution: The binomial tree for the stock price is

$$
\begin{tikzcd}
 & & S(2) = \frac{49}{16} \\
 & S(1) = \frac{7}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
S(0) = 1 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2) = \frac{7}{8} \\
 & \boxed{S(1) = \dfrac{1}{2}} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \boxed{S(2) = \dfrac{1}{4}}
\end{tikzcd}
$$

When the price of the stock in the paths above is within a box, the put option is in the money. In fact, the binomial tree for the intrinsic value $Y(t)$ of the American put is

$$
\begin{tikzcd}
 & & Y(2) = 0 \\
 & Y(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
Y(0) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(2) = 0 \\
 & Y(1) = \frac{1}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & Y(2) = \frac{1}{2}
\end{tikzcd}
$$

Now we compute the value $\hat{\Pi}_{put}(t)$ of the American put option. At time of maturity is given by the pay-off. At times $t = 0, 1$ we use the recurrence formula

$$\hat{\Pi}_{put}(t) = \max(Y(t), e^{-r}(q_u \hat{\Pi}^u_{put}(t+1)) + q_d \hat{\Pi}^d_{put}(t+1)),$$

where in this case we have $q_u = q_d = 1/2$. At time $t = 1$ we have

$$
\begin{aligned}
\hat{\Pi}_{put}(1) &= \max\left[Y(1), \frac{4}{9}(\hat{\Pi}^u_{put}(2) + \hat{\Pi}^d_{put}(2))\right] \\
&= \max\left[Y(1), \frac{4}{9}\left(\left(\frac{3}{4} - \frac{7}{4}S(1)\right)_+ + \left(\frac{3}{4} - \frac{1}{2}S(1)\right)_+\right)\right].
\end{aligned}
$$

Since

$$
Y^u(1) = \left(\frac{3}{4} - \frac{7}{4}\right)_+ = 0, \quad Y^d(1) = \left(\frac{3}{4} - \frac{1}{2}\right)_+ = \frac{1}{4},
$$

we find

$$
\hat{\Pi}^u_{put}(1) = \max[0, 0] = 0, \quad \hat{\Pi}^d_{put}(1) = \max\left[\frac{1}{4}, \frac{2}{9}\right] = \frac{1}{4}
$$

and so

$$
\hat{\Pi}_{put}(0) = \max\left[Y(0), \frac{4}{9}(\hat{\Pi}^u_{put}(1) + \hat{\Pi}^d_{put}(1))\right] = \frac{1}{9}.
$$

Hence the price of the American put corresponding to the different paths of the stock price is as follows:

$$
\begin{tikzcd}
 & & \hat{\Pi}_{put}(2) = 0 \\
 & \hat{\Pi}_{put}(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
\boxed{\hat{\Pi}_{put}(0) = \frac{1}{9}} \arrow[ur, "u"] \arrow[dr, "d"] & & \hat{\Pi}_{put}(2) = 0 \\
 & \boxed{\hat{\Pi}_{put}(1) = \frac{1}{4}} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \hat{\Pi}_{put}(2) = \frac{1}{2}
\end{tikzcd}
$$

This concludes the first part of the exercise (2 points). The hedging portfolio is computed by the formulas, for $t = 1, 2$,

$$
\hat{h}_S(t) = \frac{1}{S(t-1)} \frac{\hat{\Pi}^u_{put}(t) - \hat{\Pi}^d_{put}(t)}{e^u - e^d}, \tag{3}
$$

$$
\hat{h}_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \hat{\Pi}^d_{put}(t) - e^d \hat{\Pi}^u_{put}(t)}{e^u - e^d}. \tag{4}
$$

Hence

$$\begin{cases} h_S(2) = 0 & \text{if } S(1) = 7/4 \\ h_S(2) = -\frac{4}{5} & \text{if } S(1) = 1/2 \end{cases} \qquad h_S(1) = -\frac{1}{5}.$$

$$\begin{cases} h_B(2) = 0 & \text{if } S(1) = 7/4 \\ h_B(2) = \frac{224}{405}\frac{1}{B_0} & \text{if } S(1) = 1/2 \end{cases} \qquad h_B(1) = \frac{14}{45}\frac{1}{B_0}.$$

where $B_0 = B(0)$ is the initial value of the bond. This concludes the second part of the exercise (2 points). The put-call carity should not hold in this case, because the option is American. To verify this we compute first the fair price $\hat{\Pi}_{call}(t)$ of the American call with the same parameters of the put option; we find easily

$$\begin{tikzcd}
 & & \hat{\Pi}_{call}(2) = \frac{37}{16} \\
 & \hat{\Pi}_{call}(1) = \frac{39}{36} \arrow[ur, "u"] \arrow[dr, "d"] & \\
\hat{\Pi}_{call}(0) = \frac{41}{81} \arrow[ur, "u"] \arrow[dr, "d"] & & \hat{\Pi}_{call}(2) = \frac{1}{8} \\
 & \hat{\Pi}_{call}(1) = \frac{1}{18} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \hat{\Pi}_{call}(2) = 0
\end{tikzcd}$$

Letting $Q(t) = \hat{\Pi}_{call}(t) - \hat{\Pi}_{put}(t) - S(t) + Ke^{-r(2-t)}$, $t = 0, 1, 2$, we find easily that $Q = 0$ only at maturity and when $S(1) = 7/4$.